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Asymptotic periodicity for some evolution equations in Banach spaces

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ABSTRACT

This work deals with the existence and uniqueness of pseudo-almost periodic and asymptotically ω-periodic mild solutions to some evolution equations in Banach spaces.

1. Introduction

Pseudo-almost periodic and asymptotically periodic functions have many applications in several problems, for example in the theory of functional-differential equations, integral equations and partial differential equations. From an applied perspective asymptotically periodic systems describe our world more realistically and more accurately than periodic ones. There is much interest in developing the qualitative theory and numerical methods of such systems. Related to this subject we refer the reader to [1–3] and the references therein. The concept of pseudo-almost periodicity was introduced in the literature in the early nineties by Zhang [4–8]. Since then, it has attracted the attention of many researchers (see [9–22]). The notion of weighted pseudo-almost periodicity was introduced by Diagana [23] in 2006 and then studied in [24–27]. To construct those weighted pseudo-almost periodic functions, the main idea consists of enlarging the so-called ergodic component. See the recent paper by Agarwal et al. [25], where they discussed existence and uniqueness of a weighted pseudo-almost periodic (mild) solution to a class of semi-linear fractional differential equations. Furthermore, the authors gave applications to abstract partial evolution (respectively, fractional relaxation–oscillation) equations.

This paper is a natural continuation of the work in [28], which investigates the periodicity of evolution equations. Firstly, we study in this work sufficient conditions for the existence and uniqueness of a weighted pseudo-almost periodic (mild) solution to the following semi-linear integral equations with infinite delay of the form

$$u(t) = \int_{-\infty}^{t} a(t-s)\big[Au(s) + f(s, u(s))\big]\mathrm{d}s, \quad t \in \mathbb{R}, \tag{1.1}$$

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where $a \in L^1([0,\infty))$, $A : D(A) \subset X \to X$ is the generator of an integral resolvent family (see Definition 2.2) defined on a complex Banach space X and $f : \mathbb{R} \times X \to X$ is a weighted pseudo-almost periodic function (see Definition 2.1) satisfying suitable conditions in the second variable. We remark that equations of type (1.1) arise in the study of heat flow in materials of fading memory type (see [29,30]). The regularity of solutions for (1.1) in the space of almost automorphic (compact almost automorphic, pseudo compact almost automophic) functions was considered in [31] (resp. [32,28]). The existence of weighted pseudo-almost periodic (mild) solutions for abstract integral equations with infinite delay of type (1.1) remains an untreated topic in the literature.

Anticipating a wide interest in the subject, this paper contributes in filling this important gap. We also consider the same problem for evolution equations with a linear part dominated by a Hille–Yosida operator of negative type. Many questions in connection with this kind of equations remain unanswered. We remark that there is a great variety of semi-linear differential equations with a linear part dominated by a non-densely defined operator. Being non-dense arises, for example, from restrictions made on the space where the equation is considered and from boundary conditions (cf. [33–40]). Secondly, we use the results developed in Sections 2.3 and 2.4 to give sufficient conditions for the existence of asymptotically ω-periodic (mild) solutions to the abstract neutral integro-differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}D(t,u_t) = \int_0^t a(t-s)AD(s,u_s)\mathrm{d}s + F(t,u_t), \quad t \geq 0, \tag{1.2}$$

$$u_0 = \psi \in \mathcal{B}, \tag{1.3}$$

where the history $u_t : (-\infty, 0] \to X$, defined by $u_t(\theta) = u(t+\theta)$, belongs to some abstract space $\mathcal{B}$ defined axiomatically (see for instance [41] for applications and references); X is a Banach space; $D(t,\xi) = \xi(0) + G(t,\xi)$, $\xi \in \mathcal{B}$; and $F, G : [0,\infty) \times \mathcal{B} \to X$ are appropriate functions. Recently, Agarwal et al. [28] studied the existence of S-asymptotically ω-periodic (mild) solutions to Eq. (1.2). We remark that a vector-valued function $f \in BC([0,\infty), X)$ is called S-asymptotically ω-periodic (see [42]) if there is $\omega > 0$ such that $\lim_{t\to\infty}(f(t+\omega) - f(t)) = 0$. The literature concerning S- asymptotically ω-periodic functions with values in Banach spaces is very recent (cf. [38,43–49]). In [42] it is shown that the above property does not characterize asymptotically ω-periodic functions. To the knowledge of the authors no result yet exists for the existence of asymptotically ω-periodic (mild) solutions of (1.2)–(1.3).

We will now present a summary of this work. We have tried to make the presentation almost self-contained. Section 2 provides the definitions and preliminary results to be used in the theorems stated and proved in this article. In particular to facilitate access to the individual topics, we review in Section 2.1 some of the standard properties of weighted pseudo-almost periodic functions. In Sections 2.2 and 2.3 we present in detail the notions of integral resolvent families and phase space, respectively. In Section 2.4 we review some properties of the asymptotically ω-periodic functions. In Section 2.5 we present a new pseudo-almost periodic space introduced by Diagana and Hernández in [18], this new space is called the space of pseudo-almost periodic functions of class p. It is well known that the study of composition of two functions with special properties is very important for deep investigations. We have established a composition result for this type of functions, which is of central importance in Section 5 (see Theorems 5.17 and 5.18). In Section 2.6, we give some of the basic facts on extrapolation spaces. Abstract extrapolation spaces have been used for various purposes (see [34,50,40]). Section 3 is divided into three parts. In Section 3.1, we obtain sufficient conditions for the existence and uniqueness of a weighted pseudo-almost periodic (mild) solution to the linear equation

$$u(t) = \int_{-\infty}^t a(t-s)\big[Au(s) + f(s)\big]\mathrm{d}s, \quad t \in \mathbb{R}, \tag{1.4}$$

provided A is the generator of an integral resolvent family (see Theorem 3.3). Properties of the solutions of this linear integro-differential equation have been studied in several contexts, e.g. existence and regularity [51], maximal regularity [29], compact almost automorphy [32], pseudo compact almost automorphy [28]. In Section 3.2 (respectively, Section 3.3), we obtain very general results on the existence of weighted pseudo-almost periodic (mild) solutions to the integral equation (1.1) (respectively, semi-linear problems with non-dense domain). In Section 4, we establish sufficient conditions for the existence of asymptotically ω-periodic (mild) solutions to the neutral equations (1.2)–(1.3).

Neutral differential equations arise in many areas of applied mathematics. The literature related to ordinary neutral differential equations is quite extensive; for further information on this subject and related applications we refer the reader to [52], which contains a comprehensive presentation on those equations, see also [53–57]. We also treat the existence of asymptotically ω-periodic (mild) solutions to the following neutral integro-differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(u(t) + g(t,u(t))) = \int_0^t a(t-s)A(u(s) + g(s,u(s)))\mathrm{d}s + f(t,u(t)), \quad t \geq 0. \tag{1.5}$$

In Section 5, we give sufficient conditions for the existence of pseudo-almost periodic (mild) solutions of class p to abstract functional-differential equations with dense and non-dense domain. Finally, Section 6 is dedicated to the study of the asymptotic behavior of the solutions of a great variety of evolution equations.

We study the existence of mild solutions to our equations with the help of composition results and fixed point theory. The latter has been a very powerful and important tool in the study of nonlinear phenomena. Specifically, we use the contraction