Nonlinear Analysis 74 (2011) 1835–1840

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Common fixed point results in CAT(0) spaces

Ali Abkar *, Mohammad Eslamian

Department of Mathematics, Imam Khomeini International University, Qazvin 34149, Iran

ARTICLE INFO

Article history:
Received 28 August 2010
Accepted 28 October 2010

MSC:
47H10
47H09

Keywords:
Common fixed point
Generalized nonexpansive mapping
Multivalued mapping
CAT(0) space
Quasi-nonexpansive mapping

ABSTRACT

Let X be a complete CAT(0) space, T be a generalized multivalued nonexpansive mapping, and t be a single valued quasi-nonexpansive mapping. Under the assumption that T and t commute weakly, we shall prove the existence of a common fixed point for them. In this way, we extend and improve a number of recent results obtained by Shahzad (2009) [7,12], Shahzad and Markin (2008) [6], and Dhompongsa et al. (2005) [5].

1. Introduction

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples are pre-Hilbert spaces, $\mathbb{R}$-*trees* (see [1]), the complex Hilbert ball with a hyperbolic metric (see [2]), and many others. For more information on these spaces and on the fundamental role they play in geometry we refer the reader to [1].

Fixed point theory in CAT(0) spaces was first studied by Kirk (see [3,4]). He proved that every nonexpansive (single valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. Since then the fixed point theory for single valued and multivalued mappings in CAT(0) spaces has been rapidly developed. In 2005, Dhompongsa et al. [5] obtained a common fixed point result for commuting mappings in CAT(0) spaces. Shahzad and Markin [6] studied an invariant approximation problem and provided sufficient conditions for the existence of $z \in K \subset X$ such that $d(z, y) = \text{dist}(y, K)$ and $z = t(z) \in T(z)$ where $y \in X$, T and t are commuting nonexpansive mappings. Shahzad [7] also proved a common fixed point and invariant approximation result in a CAT(0) space in which t and T are not necessarily commuting.

In 2008, Suzuki [8] introduced a condition which is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. Suzuki's condition which was named by him the condition (C) reads as follows: A mapping T on a subset K of a Banach space X is said to satisfy the condition (C) if

$$\frac{1}{2}\|x - Tx\| \le \|x - y\| \Rightarrow \|Tx - Ty\| \le \|x - y\|, \quad x, y \in K.$$

He then proved some fixed point and convergence theorems for such mappings. Motivated by this result, Garcia-Falset et al. in [9] introduced two kinds of generalization for the condition (C) and studied both the existence of fixed points and their

* Corresponding author. Tel.: +98 9123301709; fax: +98 281 3780040.
E-mail addresses: abkar@ikiu.ac.ir, aliabkar99@gmail.com (A. Abkar), mhmdeslamian@gmail.com (M. Eslamian).

0362-546X/$ – see front matter
doi:10.1016/j.na.2010.10.056

asymptotic behavior. Very recently, the current authors used a modified Suzuki condition for multivalued mappings, and proved some fixed point theorems for multivalued mappings satisfying this condition in Banach spaces [10,11].

In this paper we consider a CAT(0) space X, together with two mappings t and T, where T belongs to the new class of generalized nonexpansive multivalued mappings (in the sense of Suzuki) and t is a single valued quasi-nonexpansive mapping. We shall establish a common fixed point for these two mappings. Our result improves a number of very recent results of Shahzad [7,12], as well as those of Shahzad and Markin [6], and of Dhompongsa et al. [5].

2. Preliminaries

Let (X, d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0, r] \subset \mathbb{R}$ to X such that $c(0) = x, c(r) = y$ and $d(c(t), c(s)) = |t - s|$ for all $s, t \in [0, r]$. In particular, the mapping c is an isometry and $d(x, y) = r$. The image of c is called a geodesic segment joining x and y which when unique is denoted by $[x, y]$. For any $x, y \in X$, we denote the point $z \in [x, y]$ such that $d(x, z) = \alpha d(x, y)$ by $z = (1 - \alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$. The space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset K of X is called convex if K includes every geodesic segment joining any two points of itself.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of points (the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom:

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\bar{x}, \bar{y} \in \overline{\triangle}$, $d(x, y) \leq d_{R^2}(\bar{x}, \bar{y})$.

The following properties of a CAT(0) space are useful (see [1]):

(i) A CAT(0) space X is uniquely geodesic.
(ii) For any $x \in X$ and any closed convex subset $K \subset X$ there is a unique closest point to $x \in K$.

A notion of $\triangle$-convergence in CAT(0) spaces based on the fact that in Hilbert spaces a bounded sequence is weakly convergent to its unique asymptotic center has been studied in [13]. Let (x_n) be a bounded sequence in X and K be a nonempty bounded subset of X. We associate this sequence with the number

$$r = r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\},$$

where

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x),$$

and the set

$$A = A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r\}.$$

The number r is known as the *asymptotic radius* of $\{x_n\}$ relative to K. Similarly, the set A is called the *asymptotic center* of $\{x_n\}$ relative to K.

In a CAT(0) space, the asymptotic center $A = A(K, \{x_n\})$ of (x_n) consists of exactly one point whenever K is closed and convex. A sequence (x_n) in a CAT(0) space X is said to be $\triangle$-convergent to $x \in X$ if x is the unique asymptotic center of every subsequence of (x_n). Notice that given $(x_n) \subset X$ such that (x_n) is $\triangle$-convergent to x and given $y \in X$ with $x \neq y$,

$$\limsup_{n\to\infty} d(x, x_n) < \limsup_{n\to\infty} d(y, x_n).$$

Thus every CAT(0) space X satisfies the Opial property.

Lemma 2.1 ([13]). *Every bounded sequence in a complete CAT(0) space has a $\triangle$-convergent subsequence.*

Lemma 2.2 ([14]). *If K is a closed convex subset of a complete CAT(0) space and if (x_n) is a bounded sequence in K, then the asymptotic center of (x_n) is in K.*

Lemma 2.3 ([15]). *Let (X, d) be a CAT(0) space. For $x, y \in X$ and $t \in [0, 1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1 - t)d(x, y).$$

We use the notation $(1 - t)x \oplus ty$ for the unique point z of the above lemma.

Definition 2.4. A point $x \in K$ is called a fixed point of T if $Tx = x$, we shall denote by $F(T)$ the set of all fixed points of T.

Definition 2.5 ([8]). Let T be a mapping on a subset K of a CAT(0) space X. T is said to satisfy condition (C) if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y), \quad x, y \in K.$$