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Global solution branches for equations involving nonhomogeneous operators of p-Laplace type

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ABSTRACT

It is shown that if μ is not an eigenvalue of an associated p-Laplacian, then the equation

$$-\operatorname{div}(\varphi(x, \nabla u)) = \mu\, |u|^{p-2}\, u + f(\lambda, x, u, \nabla u)$$

with nonhomogeneous φ (which is assumed to behave asymptotically as the function generating the associated p-Laplacian) has a global branch of solutions (λ, u). Also the case of modified p-Laplace operators and generalizations thereof are discussed.

1. Introduction

Using abstract nonlinear spectral theory and the theory of essential (0-epi) maps in the spirit of [1–3], the existence of a global branch of solutions (λ, u) for the boundary value problem

$$\begin{cases} -\Delta_p u = \mu\, |u|^{p-2}\, u + f(\lambda, x, u, \nabla u) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

was obtained in [4] when $1 < p < \infty$ (for generalizations to unbounded domains and with weight functions, see also [5,6]). Here, we denote by Δ_p either the p-Laplacian

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\, \nabla u) \tag{1.2}$$

or the modified p-Laplacian

$$\Delta_p u := \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \left(\left| \frac{\partial u}{\partial x_k} \right|^{p-2} \frac{\partial u}{\partial x_k} \right). \tag{1.3}$$

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The crucial ingredients for this result are

(1) μ is not a (nonlinear) eigenvalue of $-\Delta_p$.
(2) Δ_p induces a homeomorphism in the weak formulation of the problem.
(3) Δ_p is odd and positively homogeneous of order $p-1$.
(4) $f(0, x, \cdot, v)$ grows strictly slower than $u \mapsto |u|^{p-2}u$.

Then actually an unbounded branch of solutions will start at some point of the form $(\lambda, u) = (0, u)$. Note that in contrast to other papers which yield a global branch starting *at* the first eigenvalue λ_1 of the p-Laplacian like e.g. [7] (see also [8]) which are more in the spirit of [2,9], the mentioned result requires that μ is *not* an eigenvalue.

While most of the other hypotheses cited above are natural, one cannot expect in applications that one deals with operators which scale precisely in a positively homogeneous way, i.e. in applications the corresponding operators will usually be disturbed; similarly, one cannot expect that the operators are "precisely" odd. Concerning the term $|u|^{p-2}u$ in (1.1) such a perturbation is already "built" into the equation in terms of the map f. However, concerning Δ_p, the implicit homogeneity and oddness are severe restrictions: For real-life applications one should actually replace the power function occurring in the definition of Δ_p by a function which only "closely" behaves like a power function and is not necessarily odd.

Of course, several such approaches have been attempted. Concerning bifurcation at λ_1, the result of [7] carries over in a rather straightforward manner under such perturbations if one considers appropriate spaces [10], actually even in the "classical" $W^{1,p}$ spaces [11,12]. This is not surprising, in a sense, because this result is based on generalizations of [2], and so it is essentially only important for this result to understand how the corresponding operators behave close to $(\lambda_1, 0)$, i.e. the positive homogeneity of Δ_p actually plays no role. In contrast, for the mentioned results in [4–6], the positive homogeneity is crucial, because we will see in this paper that it is more important for these results to understand the asymptotic behaviour of the operators (i.e. "near ∞"). Of course, in the positively homogeneous case, the difference between asymptotic behaviour and local behaviour becomes invisible. For the nonhomogeneous case, the proofs given in [4–6] break down in several respects. So it is a natural question to ask whether these results were really only a particular feature of the power function or whether they actually describe a phenomenon which one can expect to occur in nature.

We will show in this paper that the latter is the case by proving that a result analogous to the above one holds under "small" perturbations of the power function, actually even for the more general boundary value problem

$$\begin{cases} -\operatorname{div}(\varphi(x, \nabla u)) = \mu\, |u|^{p-2}u + f(\lambda, x, u, \nabla u) & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\varphi(x, \cdot)\colon \mathbb{R}^N \to \mathbb{R}^N$ is not necessarily odd or positively homogeneous or radially symmetric in some sense.

In view of the above remark that only asymptotic considerations are important, it would appear natural to require that the operator J (which we will later associate to the function φ), is "asymptotically homogeneous" in the sense that

$$\lim_{t\to\infty} \frac{J(tu) - t^{p-1}J(u)}{t^q} = 0 \tag{1.5}$$

for some appropriate power q. Unfortunately, if one tries to argue along the lines of [4–6], one sees that one has to require $q \le p-1$, i.e. the best possible choice for the above condition would be $q = p-1$. Unfortunately, for each fixed u_0 and each $\lambda > 0$ the above condition with $q = p-1$ would then imply

$$\lim_{t\to\infty} \frac{J(t\lambda u_0)}{(t\lambda)^{p-1}} = \frac{J(\lambda u_0)}{\lambda^{p-1}},$$

and since the limit on the left-hand side is clearly independent of λ, this condition would actually imply

$$J(\lambda u_0) = \lambda^{p-1} J(u_0) \quad (\lambda > 0)$$

for each u_0, i.e. J is actually positively homogeneous of order $p-1$. Hence, $\varphi(x, \cdot)$ also would have to be positively homogeneous of order $p-1$ which is just the requirement which we wanted to avoid.

Therefore, we cannot use (1.5) for our considerations. Instead, we proceed in another way and require that φ approaches asymptotically a given φ_{hom} which is odd and homogeneous of order $p-1$. For example, φ_{hom} might correspond to the p-Laplacian or the modified p-Laplacian or some combination thereof. Our main point, however, is that φ itself (except for this asymptotic behaviour) need not be positively homogeneous of some order.

Under this hypothesis (and an additional hypothesis which ensures some monotonicity of the corresponding operator) we will obtain the existence of a global solution branch of (1.4), provided that μ is not an eigenvalue of the corresponding problem (1.1) with the p-Laplacian or modified p-Laplacian, respectively.

Actually, our main result extends that of [4] even in case of the classical p-Laplacian: We need less restrictive growth requirements and a much less restrictive continuity requirement on the nonlinearity f. Moreover, our conclusion about the branch is stronger.

The plan of the paper is as follows. In Section 2, we introduce an abstract result which will later give us the existence of a branch of solutions. Compared to [4], we use here some more results of degree theory avoid the hypothesis that the operators are odd. In Section 3, we associate the operator J to φ and discuss a hypothesis which will guarantee that J is a