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The existence of mild solutions for impulsive fractional partial differential equations

Xiao-Bao Shu[a,*], Yongzeng Lai[b], Yuming Chen[b]

[a] Department of Mathematics, Hunan University, Changsha 410082, PR China

[b] Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, N2L 3C5 Canada

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ABSTRACT

This paper is concerned with the existence of mild solutions for a class of impulsive fractional partial semilinear differential equations. Some errors in Mophou (2010) [2] are corrected, and some previous results are generalized.

1. Introduction

Impulsive fractional differential equations have attracted a considerable interest both in mathematics and applications since Agarwal and Benchohra published the first paper on this topic [1] in 2008; see for example [2–8]. In papers [2,3], the authors studied the existence of the mild solution for some impulsive fractional differential equations. However, in these two papers, there are two problems, (1) the definition of mild solutions given by the authors are not well defined, because classical solutions of the impulsive fractional differential equations do not satisfy the definition of a mild solution given by the authors; (2) the semigroup property $T(t+s) = T(t)T(s)$ for the system is not used correctly.

For example, consider a linear Caputo fractional differential equation

$$(D_*^{\alpha} y)(t) = -\rho y(t) + f(t), \qquad y(0) = c_1, \quad 0 < \alpha < 1. \tag{1.1}$$

Its classical solution is given by (see [9–11])

$$y(t) = c_1 E_{\alpha,1}(-\rho t^{\alpha}) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\rho(t-s)^{\alpha}) f(s) \mathrm{d}s,$$

where

$$E_{\alpha,1}(-\rho t^{\alpha}) = \frac{\rho}{\pi} \sin \pi\alpha \int_0^{\infty} \mathrm{e}^{-rt} \frac{r^{\alpha-1}}{r^{2\alpha} + 2r^{\alpha}\rho\cos\pi\alpha + \rho^2} \mathrm{d}r,$$

$$t^{\alpha-1} E_{\alpha,\alpha}(-\rho t^{\alpha}) = -\frac{1}{\pi} \sin \pi\alpha \int_0^{\infty} \mathrm{e}^{-rt} \frac{r^{\alpha}}{r^{2\alpha} + 2r^{\alpha}\rho\cos\pi\alpha + \rho^2} \mathrm{d}r.$$

Denote $T(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\rho t^{\alpha})$, $S(t) = E_{\alpha,1}(-\rho t^{\alpha})$. Then $y(t)$ can be expressed as

$$y(t) = c_1 S(t) + \int_0^t T(t-s) f(s) \mathrm{d}s, \tag{1.2}$$

* Corresponding author.
E-mail addresses: sxb0221@163.com (X.-B. Shu), ychen@wlu.ca (Y. Chen).

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where $S(t)$ and $T(t)$ are given in the above expression, and can be expressed as

$$T(t) = \frac{1}{2\pi \mathrm{i}} \int_{B_r} \mathrm{e}^{\lambda t} \frac{1}{\lambda^\alpha + \rho} \mathrm{d}\lambda, \qquad S(t) = \frac{1}{2\pi \mathrm{i}} \int_{B_r} \mathrm{e}^{\lambda t} \frac{\lambda^{\alpha-1}}{\lambda^\alpha + \rho} \mathrm{d}\lambda, \tag{1.3}$$

where B_r denotes the Bromwich path. Let $A = -\rho$. Then (1.3) can be rewritten as

$$T(t) = \frac{1}{2\pi \mathrm{i}} \int_{B_r} \mathrm{e}^{\lambda t} (\lambda^\alpha - A)^{-1} \mathrm{d}\lambda \qquad S(t) = \frac{1}{2\pi \mathrm{i}} \int_{B_r} \mathrm{e}^{\lambda t} \lambda^{\alpha-1} (\lambda^\alpha - A)^{-1}.$$

Furthermore, for system (1.1), we have $T(t) = -\frac{1}{\rho} DS(t)$. It is obvious that $T(t) \neq S(t)$ and the solution $y(t)$ of (1.1) does not satisfy Definition 3.2 in [2] and Definition 2.3 in [3], i.e., $y(t)$ is not a mild solution. On the other hand, when $0 < \alpha < 1$, the operator $T(t)$ does not satisfy the semigroup property: $T(s+t) \neq T(s)T(t)$. $T(t)$ satisfies this property only when $\alpha = 1$. In this case, we have $T(t) = S(t) = \mathrm{e}^{-\rho t}$ and $T(t+s) = T(t)T(s)$.

Remark 1.1. This example shows that a classical solution is not a mild solution based on their definitions of mild solutions. These definitions of mild solutions given by the authors are not well defined (see [12]).

Remark 1.2. Consider the Riemann–Liouville fractional differential equation

$$(D^\alpha y)(t) = -\rho y(t) + f(t), \qquad (g_{1-\alpha} * y)(0) = c_1, \quad t \geq 0,\ 0 < \alpha < 1 \tag{1.4}$$

where

$$g_{1-\alpha}(t) := \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} t^{-\alpha}, & t > 0, \\ 0, \quad t \leq 0. \end{cases}$$

Its classical solution is given by (see [13,14])

$$y(t) = c_1 E_{\alpha,\alpha}(-\rho t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\rho(t-s)^\alpha) f(s) \mathrm{d}s. \tag{1.5}$$

Let $T(t) = E_{\alpha,\alpha}(-\rho t^\alpha)$. Then we have

$$y(t) = c_1 T(t) + \int_0^t (t-s)^{\alpha-1} T(t-s) f(s) \mathrm{d}s. \tag{1.6}$$

Therefore, for system (1.4), if (1.6) holds then it must be equipped with a single initial condition, say $(g_{1-\alpha} * y)(0) = c_1$. On the other hand, for system (1.4), if we use the definition of mild solution in the literature [2], then its mild solution can be expressed as

$$y(t) = c_1 T(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s) \mathrm{d}s, \tag{1.7}$$

that is, under the initial condition $(g_{1-\alpha} * y)(0) = c_1$, the classical solution of the Riemann–Liouville fractional differential equation (1.4), does not satisfy the definition of mild solution given by the authors in [2], either. In addition, in this case, even though $T(t) = S(t) = E_{\alpha,\alpha}(-\rho t^\alpha)$, we have $T(t+s) \neq T(t)T(s)$.

In this paper, we give the definition of a mild solution, and investigate the existence of mild solutions of the system given by

$$\begin{cases} D_*^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \in I = [0, T], t \neq t_k, \\ x(0) = x_0 \in X, \\ \triangle x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, \ldots, m, \end{cases} \tag{1.8}$$

where $0 < \alpha < 1$, A is a sectorial operator on a Banach space X, D_*^α is the Caputo fractional derivative, $f : I \times X \to X$ is a given continuous function, $I_k : X \to X$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $\triangle x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$ and $x(t_k^-) = \lim_{h \to 0^-} x(t_k + h)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively.