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Solvability of second-order nonlinear three-point boundary value problems

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ABSTRACT

We are interested in the existence of nontrivial solutions to the three-point boundary value problem (BVP):

$$\begin{cases} u''(t) + f(t, u(t)) = 0, & 0 < t < 1 \\ u'(0) = 0, & u(1) = \alpha u(\eta) + \beta u'(\eta), \end{cases} \tag{$*$}$$

where $0 < \eta < 1$, $f(t, u) \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ and α, β are real constants. Fixed-point theorems and degree theory are frequently used to study such problems. Recently, the authors demonstrated that, in many situations, the shooting method proves to be an effective approach, often leading to better results with shorter proofs. Here we present another such example.

Assume that $f(t, 0) \not\equiv 0$ and that there exist nonnegative functions $k, h \in L^1(0, 1)$ such that $|f(t, w)| \le k(t)|w| + h(t)$ for all $(t, w) \in [0, 1] \times \mathbb{R}$. Sun and Liu [13] (for the special case $\beta = 0$), and Sun [14] (for the special case $\alpha = 0$) showed that, if the L^1 norm $\|k\|_1$ is sufficiently small, then there exists a nontrivial solution to the BVP $(*)$. In this paper, their results are improved using the shooting method.

1. Introduction

We are interested in the existence of nontrivial solutions to the second-order three-point boundary value problem (BVP):

$$u''(t) + f(t, u(t)) = 0, \quad 0 < t < 1 \tag{1.1}$$

$$u'(0) = 0, \quad u(1) = \alpha u(\eta) + \beta u'(\eta), \tag{1.2}$$

where $0 < \eta < 1$, α, and β are real numbers and $f \in C([0, 1] \times \mathbb{R}, \mathbb{R})$.

Second-order three-point problems were first studied by Gupta [1,2], followed by many others. In earlier work, the nonlinear function f as well as the solutions are allowed to change signs. Many later results, however, are concerned with the existence of positive solutions when $f(t, u)$ is assumed to be nonnegative, see e.g. [3–5], and of multiple solutions, see e.g. [6,7]. For more recent results on this topic, see the work of [8–10], and our papers [11,12]. In this article, we allow $f(t, u)$ to change signs but subject to linear growth in u, namely,

$$|f(t, w)| \le k(t)|w| + h(t), \tag{1.3}$$

where k, h are nonnegative functions in $L^1(0, 1)$.

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We denote, for any function $g(t, u)$,

$$g_+(t, u) = \max(g(t, u), 0),$$

and

$$g_-(t, u) = \max(-g(t, u), 0).$$

Later, we are going to use the conditions

$$\left(\frac{f(t, w)}{w}\right)_+ \le k_1(t) \tag{1.4}$$

and

$$\left(\frac{f(t, w)}{w}\right)_- \le k_2(t), \tag{1.5}$$

for w sufficiently large, which, when taken together, are weaker than (1.3). Note that we have omitted $h(t)$ in the new conditions. As we will explain later, this will not incur any loss of generality. If (1.3) holds, then (after ignoring $h(t)$) (1.4) and (1.5) hold with $k_1(t) = k_2(t) = k(t)$.

In the papers [13,14], special cases of the BVP (1.1), (1.2), with $\beta = 0$ and $\alpha = 0$, respectively, were studied, and existence criteria of solutions were established by applying the Leray–Schauder alternative fixed-point theorem. The question of uniqueness or multiplicity of solutions is not addressed. Their main results are as follows.

Theorem A ([13]). *Suppose that $f(t, 0) \not\equiv 0$ in $[0, 1]$, and that there exist nonnegative functions $k, h \in L^1(0, 1)$ such that (1.3) holds. If $\alpha \neq 1$, and*

$$\left(1 + \left|\frac{1}{1-\alpha}\right|\right) \int_0^1 (1-s)k(s)\,\mathrm{d}s + \left|\frac{\alpha}{1-\alpha}\right| \int_0^\eta k(s)\,\mathrm{d}s < 1, \tag{1.6}$$

then the BVP (1.1), (1.2) with $\beta = 0$ has a nontrivial solution.

Theorem B ([14]). *Suppose that f satisfies the assumptions in Theorem A, with (1.6) replaced by*

$$2\int_0^1 (1-s)k(s)\,\mathrm{d}s + |\beta| \int_0^\eta k(s)\,\mathrm{d}s < 1, \tag{1.7}$$

then the BVP (1.1), (1.2) with $\alpha = 0$ has a nontrivial solution.

In previous work [15,16], the authors demonstrated that three-point BVPs for second-order differential equations can often be tackled more effectively by the classical shooting method, leading to better existence criteria and shorter proofs. Another example is provided in the current paper, by showing that the shooting method can be used to improve the above-mentioned results.

Just as in our previous papers, our goal is not so much trying to get the best possible result, but rather to illustrate the methodology. In order not to mask the ideas with too much technicality, we start by stating a simpler result which already contains and extends both Theorem A and Theorem B. Its proof will be given in Section 2.

Theorem 1. *Suppose that $f(t, 0) \not\equiv 0$ in $[0, 1]$, and that there exist nonnegative functions $k, h \in L^1(0, 1)$ such that (1.3) holds, and the parameters α, β in (1.2) satisfy $\alpha \neq 1$, and*

$$\int_0^1 (1-s)k(s)\,\mathrm{d}s + \frac{|\beta|}{\max(\alpha, 1)} \int_0^\eta k(s)\,\mathrm{d}s < \begin{cases} 1, & \text{if } \alpha \le 0 \\ 1 - \min\left(\alpha, \dfrac{1}{\alpha}\right), & \text{if } \alpha > 0, \end{cases} \tag{1.8}$$

then the BVP (1.1), (1.2) has a nontrivial solution.

In Section 3, we will see how the existence criterion can be further refined by invoking the Sturm comparison theorem. In Section 4, we revisit some examples studied by Liu and Sun, and give some remarks.

When $\beta = 0$, the BVP reduces to that studied by Sun and Liu, and Theorem 1 reduces to the following.

Corollary 1. *Suppose that $f(t, 0) \not\equiv 0$ in $[0, 1]$ and that condition (1.3) holds. If k satisfies*

$$A := \int_0^1 (1-s)k(s)\,\mathrm{d}s < \begin{cases} 1, & \text{if } \alpha \le 0 \\ 1 - \min\left(\alpha, \dfrac{1}{\alpha}\right), & \text{if } \alpha > 0, \end{cases} \tag{1.9}$$

then Eq. (1.1) with the boundary conditions $u'(0) = 0$, $u(1) = \alpha u(\eta)$, $0 < \eta < 1$ has a nontrivial solution.