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Some fundamental properties for duals of Orlicz spaces

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ABSTRACT

In this paper some basic properties of Orlicz spaces are extended to their dual spaces, and finally, criteria for extreme points in these spaces are given.

1. Introduction and preliminaries

Let X be a real vector space. A function $\rho : X \to [0, +\infty]$ is called a convex modular if it satisfies the following conditions (see [1–5]):

(i) $\rho(0) = 0$ and $x = 0$ whenever $x \in X$ and $\rho(ax) = 0$ for any $a > 0$,
(ii) $\rho(-x) = \rho(x)$ for any $x \in X$,
(iii) $\rho(ax + by) \le a\rho(x) + b\rho(y)$ for all $x, y \in X$ and $a, b \ge 0$ with $a + b = 1$.

Define the modular space $X_\rho = \{x \in X : \lim_{\lambda\to 0^+} \rho(\lambda x) = 0\}$. For any $x \in X_\rho$, the Luxemburg norm is defined by

$$\|x\|_\rho = \inf\{k > 0 : \rho(k^{-1}x) \le 1\},$$

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and the Orlicz norm is defined by the Amemiya formula:

$$\|x\|_\rho^o = \inf_{k>0} k^{-1}(1+\rho(kx)).$$

Denote by $\mathbb{N}$ and R the sets of all natural numbers and real numbers, respectively. In this paper a map $\Phi : R \to [0,\infty]$ is said to be an Orlicz function if Φ is even, convex, left continuous in the extended sense (which means that infinite limits are not excluded) on $R_+ = [0,\infty)$, Φ vanishes only at 0 and Φ is not identically equal to zero (see [6,1,3,7]).

For any Orlicz function Φ, we define its complementary function $\Psi : R \to [0,\infty)$ by the formula

$$\Psi(v) = \sup_{u>0}\{u|v| - \Phi(u)\}$$

for every $v \in R$. It is well known that Ψ is also an Orlicz function, whenever $(\Phi(u)/u) \to 0$ as $u \to 0$.

By $p(u), q(v)$ and $p_-(u), q_-(v)$ we denote the right and the left derivative of $\Phi(u)$ and $\Psi(v)$, respectively. For every $u, v \in R$, we have the following Young inequality

$$uv \le |uv|\Phi(u) + \Psi(v),$$

and the equality $uv = \Phi(u) + \Psi(v)$ holds for any $v \in \mathbb{R}$ iff $u \in [q_-(v), q(v)]$ or for any $u \in \mathbb{R}$ iff $v \in [p_-(u), p(u)]$.

Let (G, Σ, μ) be a measure space with a $\sigma - \mathit{finite}$ nonatomic and complete measure μ and $L^o(\mu)$ be the space of all μ-equivalence classes of real valued and Σ-measurable functions defined on G. We say that an Orlicz function Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$ for short), if there exists a constant $K > 0$ such that $\Phi(2u) \le K\Phi(u)$ for all $u \ge 0$ if $\mu(G) = \infty$ (resp., for every $u \ge u_0$ and some $u_0 > 0$ with $\Phi(u_0) < \infty$ if $\mu(G) < \infty$). Given and Orlicz function Φ we define a convex function $I_\Phi : L^o(\mu) \to [0,\infty]$ (see [8,6,1–5,9,7]) by

$$I_\Phi(x) = \int_G \Phi(x(t))\mathrm{d}\mu(t).$$

It is clear that I_Φ is a convex modular on $L^0(\mu)$. Then the Orlicz function space L_Φ and its subspace E_Φ are defined as follows:

$$L_\Phi = \{x \in L^o(\mu) : I_\Phi(cx) < \infty \text{ for some } c > 0\},$$
$$E_\Phi = \{x \in L^o(\mu) : I_\Phi(cx) < \infty \text{ for any } c > 0\}.$$

Denote by $\|\cdot\|_\Phi, \|\cdot\|_\Phi^o$ the Luxemburg norm and the Orlicz norm on the space L_Φ, respectively. To simplify notations, we put $L_\Phi = (L_\Phi, \|\cdot\|_\Phi)$ and $L_\Phi^o = (L_\Phi, \|\cdot\|_\Phi^o)$. Let us note that Orlicz [9] defined the Orlicz norm on L_Φ by use of another formula

$$\|x\|_\Phi^o = \sup\left\{\int_G x(t)y(t)\mathrm{d}\mu(t) : I_\Psi(y) \le 1\right\}.$$

Hudzik and Maligranda have proved in [10] that the above norm coincides in L_Φ with the norm defined by the Amemiya formula for any Orlicz function Φ.

Let Φ be an Orlicz function. An interval $[a,b]$ is called a structural affine interval of Φ, or simply, SAI of Φ, provided that Φ is affine on $[a,b]$ and it is not affine on either $[a-\varepsilon, b]$ or $[a, b+\varepsilon]$ for any $\varepsilon > 0$. Let $\{[a_i, b_i]\}_i$ be the sequence of all SAIs of Φ. We call the set

$$S_\Phi = R \setminus \left[\bigcup_i (a_i, b_i)\right]$$

as the set of strictly convex points of Φ. Let us note that $R \setminus S_\Phi$ is the union of at most countably many intervals.

Let $B(X)$ and $S(X)$ be the unit ball and the unit sphere of a Banach space X, respectively. A point $x \in B(X)$ is called an extreme point of $B(X)$ if $2x = y + z$ and $y, z \in B(X)$ imply $y = z$. The set of all extreme points of $B(X)$ is denoted by $\operatorname{ext} B(X)$. It is evident that $\operatorname{ext} B(X) \subset S(X)$. We denote by L_Φ^* the dual space of L_Φ and $\varphi \in L_\Phi^*$ is called a singular functional or simply, $\varphi \in F$, if $\varphi(E_\Phi) = \{0\}$.

Let us recall that in [11] global smoothness, non-squareness and rotundity properties of the duals and the biduals of Orlicz spaces have been considered.

The following result is due to Ando (see [12]), who proved the result for N-functions Φ (that is for finitely valued Orlicz functions Φ which satisfy $\Phi(u)/u \to 0$ as $u \to 0$ and $\Phi(u)/u \to \infty$ as $u \to \infty$) only. However, the result is also true for arbitrary Orlicz function Φ by the result of Maligranda and Wnuk [2] who proved that the Köthe dual of the Orlicz space L_Φ is the Orlicz space L_Ψ^0, where Ψ is the function complementary to Φ in the sense of Young, as well as by the general result which says that the topological dual of any Köthe space E is equal to the direct sum of the Köthe dual E' and the singular dual of E (see [13]).

Lemma 1.1. *Any functional $f \in L_\Phi^*$ has a unique decomposition*

$$f = v + \varphi \quad (v \in L_\Psi, \varphi \in F), \tag{1.1}$$

where v means in fact the regular functional defined by the function v on L_Φ by the formula $\langle v, x\rangle = \int_G v(t)x(t)\mathrm{d}\mu(t)$ for any $x \in L_\Phi$.