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On positive solutions of an m-point nonhomogeneous singular boundary value problem

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ABSTRACT

This paper is concerned with the existence, nonexistence and multiplicity of positive solutions for the following second order m-point nonhomogeneous singular boundary value problem

$$u''(t) + a(t)f(t, u) = 0, \quad t \in (0, 1),$$
$$u(0) = 0, \qquad u(1) - \sum_{i=1}^{m-2} k_i u(\xi_i) = b,$$

where $b > 0$, $k_i > 0\,(i = 1, 2, \ldots, m-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\sum_{i=1}^{m-2} k_i\xi_i < 1$, $a(t)$ may be singular at $t = 0$ and/or $t = 1$. We show that, under suitable conditions, there exists a positive number b^* such that the above problem has at least two positive solutions for $0 < b < b^*$, at least one positive solution for $b = b^*$ and no solution for $b > b^*$ by using the Krasnosel'skii–Guo fixed point theorem, the upper–lower solutions method and topological degree theory.

1. Introduction

The study of multi-point boundary value problems (BVPs) for second order differential equations was initiated by Bitsadze and Samarski [1] and later continued by Il'in and Moiseev [2,3] and Gupta [4]. Since then, great efforts have been devoted to nonlinear multi-point BVPs due to their theoretical challenge and great application potential. Many results on the existence of positive solutions for multi-point BVPs have been obtained, and for more details the reader is referred to [5–11] and the references therein. Recently, Ma [8] studied the second order differential equation

$$u''(t) + a(t)f(u) = 0, \quad t \in (0, 1), \tag{1.1}$$

subject to the nonhomogeneous three-point boundary conditions (BCs)

$$u(0) = 0, \qquad u(1) - \alpha u(\eta) = b,$$

where $b > 0$, $\alpha > 0$, $0 < \eta < 1$, $\alpha\eta < 1$, and $a(t)$ is continuous on $[0, 1]$. However, Ma only established the existence of one positive solution and the nonexistence of solutions for the above BVP in which f is superlinear by applying the Schauder fixed-point theorem. The main result of [8] was subsequently extended by Ma in [9] to more general second

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order nonhomogeneous m-point BVPs. More recently, second order differential equation (1.1) was studied with n-point nonhomogeneous BCs

$$u(0)=0, \qquad u(1)-\sum_{i=1}^{n-2} k_i u(\xi_i)=b \tag{1.2}$$

in [10], where $b \geq 0$, $k_i > 0$ $(i=1,2,\ldots,n-2)$, $0<\xi_1<\xi_2<\cdots<\xi_{n-2}<1$, $\sum_{i=1}^{n-2} k_i\xi_i < 1$, and $a(t)$ is continuous on $[0,1]$. Chen et al. [10] generalized the results of [9] and established the existence of one positive solution and the nonexistence of solutions for the BVP (1.1) and (1.2) in which f is superlinear or sublinear by applying the Krasnosel'skii–Guo fixed point theorem. Paper [11] has studied Eq. (1.1) with the BC $u(0)=0$ in (1.2) replaced by $u'(0)=0$, here $\sum_{i=1}^{n-2} k_i < 1$, some sufficient conditions guaranteeing the existence of one positive solution are obtained.

Motivated by the above works, we consider the following more general second order m-point nonhomogeneous singular BVP

$$\begin{cases} u''(t)+a(t)f(t,u)=0, \quad t\in(0,1), \\ u(0)=0, \qquad u(1)-\sum_{i=1}^{m-2} k_i u(\xi_i)=b, \end{cases} \tag{P_b}$$

where $b>0$, $k_i>0$ $(i=1,2,\ldots,m-2)$, $0<\xi_1<\xi_2<\cdots<\xi_{m-2}<1$, $\sum_{i=1}^{m-2} k_i\xi_i<1$; $a(t)$ may be singular at $t=0$ and/or $t=1$. Under certain suitable conditions, we establish the results of existence, nonexistence and multiplicity of positive solutions of the problem. The results obtained significantly improve and generalize the results of [8,10], and the method used for the analysis is significantly different from [8–11]. The main result of the present paper is summarized as follows.

Theorem 1.1. *Suppose the following conditions hold:*

(H_1) $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ *is continuous.*
(H_2) $a:(0,1)\to[0,+\infty)$ *is continuous,* $a(t)\not\equiv 0$ *on any subinterval of* $(0,1)$ *and* $L=\int_0^1 G(s,s)a(s)\mathrm{d}s<+\infty$, *where* $G(s,s)$ *is given in Section* 2.
(H_3) $f_0=\lim_{u\to 0^+}\max_{0\leq t\leq 1}\frac{f(t,u)}{u}=0$, $f_\infty=\lim_{u\to+\infty}\min_{\xi_1\leq t\leq\xi_{m-2}}\frac{f(t,u)}{u}=+\infty$.

Then there exists a positive number b^* such that problem (P_b) has at least two positive solutions for $0<b<b^*$, at least one positive solution for $b=b^*$ and no solution for $b>b^*$.

The proof of the above theorem is based on the Krasnosel'skii–Guo fixed point theorem, the upper–lower solutions method and topological degree theory and was motivated by [12,13].

Since this paper was submitted some other relevant papers have been published, see, for example, [14–19]. In [14], Kwong and Wong studied the second order multi-point BVPs with a nonhomogeneous BC at the right endpoint. For the nonsingular case, the existence results of positive solutions were shown by using Shooting Methods. Liu [15] has used the five functionals fixed point theorem to establish results of at least three positive solutions for nonhomogeneous multi-point BVPs of second order differential equations with the one-dimensional p-Laplacian. In [16], by employing the Krasnosel'skii–Guo fixed point theorem and Schauder's fixed point theorem, Sun studied the existence and nonexistence of positive solutions to the third order three-point nonhomogeneous BVP. The higher order multi-point BVPs with nonhomogeneous BCs were considered in [17,18]. The $2n$th order BVP was studied by Kong and Kong [17] who showed the existence, nonexistence and multiplicity of positive solutions by using the fixed point index theory, the Schauder fixed point theorem, and the lower and upper solutions method. In [18], criteria for the existence of positive solutions of nth order BVPs are established via the Krein–Rutman theorem and fixed point index theory. The existence of multiple positive solutions for more general nonlocal BVPs of arbitrary order was studied by Webb and Infante in [19], and nonhomogeneous nonlocal BVPs also have been studied.

2. Preliminary lemmas

Let $E=C[0,1]$, then E is a Banach space with the norm $\|u\|=\sup_{t\in[0,1]}|u(t)|$. A function u is said to be a solution of the m-point BVP (P_b) if $u\in C[0,1]\cap C^2(0,1)$ satisfies the BVP (P_b). In addition, the solution u of the BVP (P_b) is said to be a positive solution if $u(t)>0$ for $t\in(0,1)$.

Lemma 2.1. *Suppose* $0<\sum_{i=1}^{m-2} k_i\xi_i<1$. *If* $y(t)\in C(0,1)$ *with* $\int_0^1 G(s,s)y(s)\mathrm{d}s<\infty$, *then the Green's function for the homogeneous BVP*

$$u''(t)+y(t)=0, \quad t\in(0,1), \tag{2.1}$$

$$u(0)=0, \qquad u(1)-\sum_{i=1}^{m-2} k_i u(\xi_i)=0 \tag{2.2}$$