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Compactness by the Hausdorff measure of noncompactness

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ABSTRACT

In the present paper, we establish some identities or estimates for the operator norms and the Hausdorff measures of noncompactness of certain operators given by infinite matrices that map an arbitrary *BK*-space into the sequence spaces c_0, c, ℓ_∞ and ℓ_1, and into the matrix domains of triangles in these spaces. Furthermore, by using the Hausdorff measure of noncompactness, we apply our results to characterize some classes of compact operators on the *BK*-spaces.

1. Background, notations and preliminaries

Let X be a normed space. Then, we write S_X and $\bar{B}_X$ for the unit sphere and the closed unit ball in X, that is, $S_X = \{x \in X : \|x\| = 1\}$ and $\bar{B}_X = \{x \in X : \|x\| \leq 1\}$. If X and Y are *Banach spaces*, then $\mathcal{B}(X, Y)$ denotes the set of all bounded (continuous) linear operators $L : X \to Y$, which is a *Banach space* with the operator norm given by $\|L\| = \sup_{x\in S_X} \|L(x)\|_Y$ for all $L \in \mathcal{B}(X, Y)$. A linear operator $L : X \to Y$ is said to be *compact* if the domain of L is all of X and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a subsequence which converges in Y. We denote the class of all compact operators in $\mathcal{B}(X, Y)$ by $\mathcal{C}(X, Y)$. An operator $L \in \mathcal{B}(X, Y)$ is said to be *of finite rank* if $\dim R(L) < \infty$, where $R(L)$ is the range space of L. An operator of finite rank is clearly compact.

By w, we shall denote the space of all complex sequences. If $x \in w$, then we write $x = (x_k)$ instead of $x = (x_k)_{k=0}^{\infty}$. Also, we write ϕ for the set of all finite sequences that terminate in zeros. Further, we use the conventions that $e = (1, 1, \ldots)$ and $e^{(k)}$ is the sequence whose only non-zero term is 1 in the kth place for each $k \in \mathbb{N}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Any vector subspace of w is called a *sequence space*. We shall write ℓ_∞, c and c_0 for the sequence spaces of all bounded, convergent and null sequences, respectively. Further, by ℓ_1 and ℓ_p $(1 < p < \infty)$, we denote the sequence spaces of all absolutely and p-absolutely convergent series, respectively. Furthermore, we write bs, cs and cs_0 for the sequence spaces of all bounded, convergent and null series, respectively. Moreover, we denote the space of all sequences of bounded variation by bv, that is, $bv = \{x = (x_k) \in w : (x_k - x_{k-1}) \in \ell_1\}$. By the *classical sequence spaces*, we mean the spaces c_0, c, ℓ_∞ and ℓ_p $(1 \leq p < \infty)$.

The α-, β- and γ-*duals* of a subset X of w are respectively defined by

$$X^\alpha = \{a = (a_k) \in w : ax = (a_k x_k) \in \ell_1 \text{ for all } x = (x_k) \in X\},$$

$$X^\beta = \{a = (a_k) \in w : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X\}$$

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and

$$X^{\gamma} = \left\{a = (a_k) \in w : ax = (a_k x_k) \in bs \text{ for all } x = (x_k) \in X\right\}.$$

Throughout this paper, the matrices are infinite matrices of complex numbers. If A is an infinite matrix with complex entries a_{nk} $(n, k \in \mathbb{N})$, then we write $A = (a_{nk})$ instead of $A = (a_{nk})_{n,k=0}^{\infty}$. Also, we write A_n for the sequence in the nth row of A, i.e., $A_n = (a_{nk})_{k=0}^{\infty}$ for every $n \in \mathbb{N}$. In addition, if $x = (x_k) \in w$, then we define the *A-transform* of x as the sequence $Ax = (A_n(x))_{n=0}^{\infty}$, where

$$A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k; \quad (n \in \mathbb{N}) \tag{1.1}$$

provided the series on the right converges for each $n \in \mathbb{N}$.

Let X and Y be subsets of w and $A = (a_{nk})$ an infinite matrix. Then, we say that A defines a *matrix mapping* from X into Y, and we denote it by writing $A : X \to Y$, if Ax exists and is in Y for all $x \in X$. By (X, Y), we denote the class of all infinite matrices that map X into Y. Thus $A \in (X, Y)$ if and only if $A_n \in X^{\beta}$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

For any subset X of w, the *matrix domain* of an infinite matrix A in X is defined by

$$X_A = \left\{x \in w : Ax \in X\right\}.$$

The theory of FK- and BK-spaces is the most powerful tool in the characterization of matrix transformations between sequence spaces.

A sequence space X is called an *FK-space* if it is a complete linear metric space with continuous coordinates $p_n : X \to \mathbb{C}$ $(n \in \mathbb{N})$, where $\mathbb{C}$ is the complex field and $p_n(x) = x_n$ for all $x = (x_k) \in X$ and every $n \in \mathbb{N}$. A *BK-space* is a normed FK-space [1], that is, a *BK-space* is a Banach sequence space with continuous coordinates [2].

The classical sequence spaces are BK-spaces with their natural norms [2, Example 1.1]. More precisely, the spaces c_0, c and ℓ_∞ are BK-spaces with the usual sup-norm given by $\|x\|_{\ell_\infty} = \sup_k |x_k|$, where the supremum is taken over all $k \in \mathbb{N}$. Also, the space ℓ_p is a BK-space with the usual ℓ_p-norm defined by $\|x\|_{\ell_p} = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$, where $1 \le p < \infty$.

A sequence $(b_n)_{n=0}^{\infty}$ in a linear metric space X is called a *Schauder basis* for X if for every $x \in X$ there is a unique sequence $(\alpha_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \alpha_n b_n$, that is, $\lim_{m\to\infty}(\sum_{n=0}^{m} \alpha_n b_n) = x$ [3, Definition 1.6].

Although the space ℓ_∞ has no Schauder basis, the spaces c_0, c and ℓ_p $(1 \le p < \infty)$ all have Schauder bases [3, Theorem 1.10].

An FK-space $X \supset \phi$ is said to have AK if every $x = (x_k) \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is, $x^{[m]} = \sum_{k=0}^{m} x_k e^{(k)} \to x$ as $m \to \infty$ [4]. Here, $x^{[m]}$ is called the *m-section* of x $(m \in \mathbb{N})$.

Among the other classical sequence spaces, the spaces c_0 and ℓ_p $(1 \le p < \infty)$ have AK [5, Example 3.16].

If $X \supset \phi$ is a BK-space and $a = (a_k) \in w$, then we define

$$\|a\|_X^* = \sup_{x\in S_X} \left|\sum_{k=0}^{\infty} a_k x_k\right| \tag{1.2}$$

provided the expression on the right hand side exists and is finite [6], which is the case whenever $a \in X^{\beta}$ [7, Theorem 7.2.9].

Throughout, let $1 \le p \le \infty$ and q denote the conjugate of p, that is, $q = \infty$ for $p = 1$, $q = p/(p-1)$ for $1 < p < \infty$ and $q = 1$ for $p = \infty$.

The following known results are fundamental for our investigation, and we may begin with the α-, β- and γ-duals of the classical sequence spaces.

Lemma 1.1 ([5, Example 4.4]). *Let † denote any of the symbols α, β or γ. Then, we have $c_0^{\dagger} = c^{\dagger} = \ell_\infty^{\dagger} = \ell_1$, $\ell_1^{\dagger} = \ell_\infty$ and $\ell_p^{\dagger} = \ell_q$, where $1 < p < \infty$ and $q = p/(p-1)$.*

Lemma 1.2 ([3, Theorem 1.29]). *Let X be any of the spaces c_0, c, ℓ_∞ or ℓ_p $(1 \le p < \infty)$. Then, we have $\|\cdot\|_X^* = \|\cdot\|_{X^\beta}$ on X^{β}, where $\|\cdot\|_{X^\beta}$ denotes the natural norm on the dual space X^{β}.*

Lemma 1.3 ([8, Lemma 15(a), (b)]). *Let $X \supset \phi$ and Y be BK-spaces. Then, we have*

(a) *$(X, Y) \subset \mathcal{B}(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ by $L_A(x) = Ax$ for all $x \in X$.*
(b) *If X has AK, then $\mathcal{B}(X, Y) \subset (X, Y)$, that is, for every operator $L \in \mathcal{B}(X, Y)$ there exists a matrix $A \in (X, Y)$ such that $L(x) = Ax$ for all $x \in X$.*

Furthermore, we have the following results on the operator norms.

Lemma 1.4 ([9, Lemma 5.2]). *Let $X \supset \phi$ be a BK-space and Y be any of the spaces c_0, c or ℓ_∞. If $A \in (X, Y)$, then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$