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Well-posedness and existence of bound states for a coupled Schrödinger-gKdV system

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ABSTRACT

We derive some new results concerning the Cauchy problem and the existence of bound states for a class of coupled nonlinear Schrödinger-gKdV systems. In particular, we obtain the existence of strong global solutions for initial data in the energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R})$, generalizing previous results obtained in Tsutsumi (1993) [11], Corcho and Linares (2007) [13] and Dias et al. (submitted for publication) [14] for the nonlinear Schrödinger-KdV system.

1. Introduction

The generalized KdV (gKdV) equation

$$v_t + v_{xxx} + a(v)v_x = 0, \quad a \in C^\infty(\mathbb{R}), \tag{1}$$

was introduced in [1] as a model for the propagation of nonlinear waves in an anharmonic lattice. In recent years, many authors treated the case where the function a is a polynomial: see for instance [2,3] for well-posedness, [4] for the stability of solitary waves and [5–7] for the study of the dispersion rates of the solutions to (1). In contrast, few results exist in the literature concerning more general nonlinearities (see on this subject [8–10]).

In the present paper, we will consider the coupled system

$$\begin{cases} iu_t + u_{xx} = \alpha uv + \beta |u|^q u & \text{(a)} \\ v_t + v_{xxx} + a(v)v_x = \gamma(|u|^2)_x & \text{(b)} \end{cases} \tag{2}$$

with initial data

$$\begin{aligned} &u(x,0) = u_o(x), \\ &v(x,0) = v_o(x), \quad x \in \mathbb{R}, \end{aligned} \tag{3}$$

where α, β and γ are real constants, $1 < q < 4$, u is a complex-valued function, v is a real-valued function and a is a C^∞ real-valued function satisfying some growth conditions that will be explained later.

This system is a natural extension of the coupled nonlinear Schrödinger–KdV system

$$\begin{cases} iu_t + u_{xx} = \alpha uv + \beta |u|^q u \\ v_t + v_{xxx} + vv_x = \gamma(|u|^2)_x \end{cases} \tag{4}$$

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first studied in [11] and then in [12,13] for the Cauchy problem, and in [14] for the existence of positive bound states when $q = 2$. This class of coupled systems, namely when the second equation is a linear (or quasilinear) transport equation, was introduced by Benney in [15] as a universal model for the interaction of short and long waves and was fully studied by several authors in [16–21].

This paper is organized as follows:

In Section 2 we extend the results obtained in [11] concerning the Cauchy problem in the case $a(v) = v$ for initial data $(u_o, v_o) \in H^{\frac{5}{2}}(\mathbb{R}) \times H^2(\mathbb{R})$. By an adaptation of the proofs in [11] we obtain (see Theorem 2.2) a global existence and uniqueness result.

In Section 3, by a regularization method, we extend the global existence obtained in the previous section for initial data in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. In [13] very sharp results on the Cauchy problem associated with (4) were obtained, but the techniques employed do not seem suitable to handle the more general nonlinearity $a(v)v_x$ in (2).

Finally, in Section 4, we extend to (2) the results obtained in [14] for the Schrödinger–KdV system (4) concerning the existence of positive non-trivial bound state solutions.

We will study the system

$$\begin{cases} -\phi'' + c^*\phi = -\beta|\phi|^q\phi - \alpha\phi\psi \\ -\psi'' + c\psi = (k+2)\psi^{k+1} - \dfrac{\alpha}{2}\phi^2. \end{cases} \tag{5}$$

In [22], the existence of solutions for a system similar to (5) is derived in the special case where $q = 2$ and $k = 2$. However the method employed cannot be exploited for $k = 1$.

Here, for $1 < q < 4$ and $k = 1, 2, 3$, we will obtain a two-parameter family of solutions to (5), smooth, positive, radially decreasing and with exponential decay at infinity.

We finish this section by introducing a few notations:

In what follows, $\{U_S(t)\}_{t\in\mathbb{R}}$ and $\{U_K(t)\}_{t\in\mathbb{R}}$ will be, respectively, the free Schrödinger and KdV evolution groups defined by

$$U_S(t) = e^{it\frac{\partial^2}{\partial x^2}} = \mathcal{F}_x^{-1} e^{-it\xi^2} \mathcal{F}_x, \tag{6}$$

and

$$U_K(t) = e^{-t\frac{\partial^3}{\partial x^3}} = \mathcal{F}_x^{-1} e^{it\xi^3} \mathcal{F}_x, \quad t \in \mathbb{R}. \tag{7}$$

Here, $\mathcal{F}_x$ and $\mathcal{F}_x^{-1}$ denote the Fourier and the inverse Fourier transform in x. Also, for $s \geq 0$, we put

$$D_x^s = \mathcal{F}_x^{-1} |\xi|^s \mathcal{F}_x.$$

Finally, for $1 \leq p, q \leq +\infty$ and $T > 0$, we will consider the spaces $L^p(\mathbb{R};\ L^q(]-T,T[))$ and $L^q(]-T,T[;L^p(\mathbb{R}))$ endowed respectively with the norms

$$\|f\|_{L_x^p L_T^q} = \left(\int_{\mathbb{R}} \left(\int_{-T}^{T} |f(x,t)|^q dt\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_{L_T^q L_x^p} = \left(\int_{-T}^{T} \left(\int_{\mathbb{R}} |f(x,t)|^p dx\right)^{\frac{q}{p}} dx\right)^{\frac{1}{q}}.$$

2. Global existence for initial data in $H^{\frac{5}{2}}(\mathbb{R}) \times H^2(\mathbb{R})$

Following [11], we introduce the function space

$$W_T^2(\mathbb{R} \times [-T,T]; \mathbb{K}) = \{f : \mathbb{R} \times [-T,T] \to \mathbb{K};\ \partial_x^{k-1} f \in L_x^2 L_T^\infty \text{ and } \partial_x^{k+1} f \in L_x^\infty L_T^2,\ k = 1, 2\},$$

$\mathbb{K} = \mathbb{R}, \mathbb{C}$, which is a Banach space when endowed with the norm

$$\|f\|_{W_T^2} = \sum_{k=1}^{2} \|\partial_x^{k-1} f\|_{L_x^2 L_T^\infty} + \|\partial_x^{k+1} f\|_{L_x^\infty L_T^2}.$$

Furthermore, we set

$$X_T^2 = C([-T,T]; H^{\frac{5}{2}}(\mathbb{R};\mathbb{C}) \cap W_T^2(\mathbb{R} \times [-T,T];\mathbb{C})),$$
$$Y_T^2 = C([-T,T]; H^2(\mathbb{R}) \cap W_T^2(\mathbb{R} \times [-T,T];\mathbb{R})),$$
$$Z_T^2 = X_T^2 \times Y_T^2,$$