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Nonlinear functional differential equations of monotone-type in Hilbert spaces[☆]

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ABSTRACT

In this paper, we study a class of semilinear functional evolution equations with finite delay in which the nonlinearity satisfies the weak condition of demicontinuity with respect to functional variable and also a semimonotone condition. We will prove the existence, uniqueness and measurability of the mild solutions based on an extension of the analogous results for non-delay initial value problems on Banach spaces and a version of recently developed random Schauder's fixed point theorem. This will be done first for the generalized solutions and then we follow an approximating procedure to obtain the same results for the mild solutions.

1. Introduction

While the standard existence and uniqueness theorems for the initial value (data) problem $\frac{dx}{dt} = f(t, x(t))$ (resp. $= f(t, x_t)$) with $x(0) = v$ (resp. $x_0 = \phi$) on a real Hilbert space H, assume that f satisfies a locally Lipschitz condition so that one can apply Picard's successive approximations, it is well-known ([1], problem 5, page 287) that when H is infinite dimensional, the continuity of f is not sufficient to ensure the existence of even a local solution. Nevertheless, extending the existence, uniqueness and stability results for classical ordinary [2] and functional differential equations [3] to equations in infinite dimensional spaces has been investigated by many authors among them we may point out [4–10]. In particular, based on the general results on monotone nonlinear operator equations in Hilbert and reflexive Banach spaces, Browder [4,5] proved the existence and uniqueness of generalized and also mild solutions, in the case f is continuous, carries bounded sets of $[0, T] \times H$ into bounded subsets of H and has a semimonotone property. In [10], Vainberg dropped the coercivity condition that was essential in the abstract machinery of Browder and assuming only that f is demicontinuous and bounded, established the existence and uniqueness of generalized and mild solutions. Fitzgibbon [7] discussed the problem for semilinear functional evolution equations where the associated homogeneous linear equation generates a compact evolution system and the nonlinear term is continuous and bounded. In this way, he could apply Schauder's fixed point theorem. As for the problem in the variational setting, Caraballo [11] developed a theory for the existence and stability of the strong solutions for nonlinear functional differential equations in the framework of Gelfand triple where the nonlinearity is divided into two parts, one corresponds to a family of monotone nonlinear operators satisfying some kind of coercivity condition and the other part is Lipschitz continuous. This program followed in [12] for stochastic functional evolution equations involving more general nonlinear operators.

Long-time behaviour of solutions has been also studied by some authors for both deterministic and stochastic equations [3,13–16]. Specifically, following the same method as in [17,14,18], Jahanipur [15] proved the exponentially asymptotic

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stability of the mild solutions for semilinear stochastic delay evolution equations with monotone nonlinearity. It is important in this regard that the conditions should guarantee the global existence of solutions.

When the evolution with respect to the time of a dynamic phenomenon is governed by some sort of randomness or probabilistic effects, the solution of the differential equation modeling its behaviour, depends not only on the time, but also on a variable that comes from an appropriate probability space. Therefore, we should take into account the measurability of the solutions. In this case, using the familiar method of regularization by mollifiers, Zangeneh [19,20] and Hamedani and Zangeneh [21] studied the measurability of the mild solutions of monotone evolution equations without any delay, by obtaining them as the uniform limit of a family of approximating solutions corresponding to a Lipschitz-type system of equations. On the other hand, some authors preferred to apply the tools of random fixed point theory and random operator equations [22–24] to obtain directly the measurable solutions in an abstract setting.

In this paper, we will establish the existence, uniqueness and measurability of the generalized and mild solutions of functional differential equations of monotone-type. Our novelty is mainly in the conditions we will impose on the nonlinearity f and in the method we will use to prove the measurability which is, in fact, based upon a version of random Schauder's fixed point theorem [23]. Hence, the results of this paper are extensions to functional equations with delay of the similar results obtained for example in [4,5,21,10,19,20].

2. Preliminaries

Let H be a real separable Hilbert space with the norm and inner product denoted by $\|.\|$ and $\langle ., .\rangle$ respectively. For a fixed real $r > 0$, let $C_H = C(-r, 0; H)$ be the Banach space of all continuous H-valued functions $\psi : [-r, 0] \to H$ defined on the finite delay interval $[-r, 0]$ with the usual sup-norm $\|\psi\|_{C_H} = \sup_{\theta\in[-r,0]} \|\psi(\theta)\|$. Let T be a positive real number. Given a continuous function $x : [-r, T] \to H$, we denote by $x_t \in C_H$ the function defined on $\theta \in [-r, 0]$ by $x_t(\theta) = x(t + \theta)$, for every $0 \le t \le T$. Consider the initial data problem

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = f(t, x_t), & t \in [0, T] \\ x(\theta) = \phi(\theta), & \theta \in [-r, 0], \end{cases} \tag{2.1}$$

where $\phi \in C_H$ and $f : [0, T] \times C_H \to H$. Here and elsewhere in this paper, the one-sided derivatives are meant at the end points of the interval $[0, T]$. Along with (2.1), we consider the integral equation

$$x_0 = \phi, \qquad x(t) = \phi(0) + \int_0^t f(s, x_s)\mathrm{d}s, \quad t \in [0, T]. \tag{2.2}$$

Let $f : [0, T] \times C_H \to H$ be continuous. If x is a continuous solution of (2.2), then it is easy to see that the function $t \mapsto x_t$ is also continuous from $[0, T]$ into C_H; so x is differentiable on $[0, T]$ and satisfies (2.1). Since the converse is obviously true, we conclude that in this case, problem (2.1) is equivalent to integral equation (2.2). If we drop the continuity of f, these two problems need not be equivalent. For this reason

Definition 2.1. A continuous function $x : [-r, T] \to H$ satisfying integral equation (2.2) is called a generalized solution of (2.1).

The following are the relevant hypotheses on nonlinear part f of the equation.

Hypothesis 2.2. (a) The function $f : [0, T] \times C_H \to H$ is semimonotone with parameter $M \ge 0$. By this, we mean that

$$\langle f(t, \psi_1) - f(t, \psi_2), \psi_1(0) - \psi_2(0)\rangle \le M\|\psi_1(0) - \psi_2(0)\|^2,$$

for all $t \in [0, T]$ and $\psi_1, \psi_2 \in C_H$;

(b) There exists a continuous function $h : [0, T] \times [0, \infty) \to [0, \infty)$ such that $u \mapsto h(t, u)$ is monotone increasing and $\|f(t, \psi)\| \le h(t, \|\psi\|_{C_H})$ for all $t \in [0, T]$ and $\psi \in C_H$;

(c) For each $t \in [0, T]$, the mapping $\psi \mapsto f(t, \psi)$ is demicontinuous; that is, whenever $\{\psi_n\}$ is a sequence which is strongly convergent to ψ in C_H, then $f(t, \psi_n)$ converges weakly to $f(t, \psi)$ in H.

Note that hypotheses 2.2 are more general than the corresponding hypotheses on nonlinear ordinary differential equations in Banach and Hilbert spaces [4,5,10] in that we have not only replaced the famous Lipschitz property by monotone condition, but also imposed the weak condition of demicontinuity on nonlinear term f. In Section 3, we prove that the conditions in Hypothesis 2.2 are sufficient to ensure the existence of a unique generalized solution for (2.1) on $[0, T]$.

A family $\{U(t, s) : 0 \le s \le t \le T\}$ of bounded linear operators on H is said to be an *evolution operator* if the operator-valued function $(t, s) \mapsto U(t, s)$ is strongly continuous for $0 \le s \le t \le T$ and

$$U(t, t) = I, \qquad U(t, r)U(r, s) = U(t, s), \quad 0 \le s \le r \le t \le T,$$

where I is the identity operator on H. Let $\{A(t) : 0 \le t \le T\}$ be a family of closed densely defined linear operators on H.