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A least squares Galerkin–Petrov nonconforming mixed finite element method for the stationary Conduction–Convection problem☆

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ABSTRACT

In this paper, a least squares Galerkin–Petrov nonconforming mixed finite element method (LSGPNMFM) is proposed and analyzed for the stationary Conduction–Convection problem. We use P_2-nonconforming as approximation space for the velocity, the linear element for the pressure space and the quadratic element for the temperature space. The mixed finite element spaces X_h and M_h need not satisfy inf–sup condition, the existence, uniqueness and convergence of the discrete solution are presented and error estimates of optimal order are derived in the case of sufficient viscosity.

1. Introduction

We consider the following stationary Conduction–Convection problem (cf. [1–4]):
Problem (I) Find $u = (u^1, u^2)$, p and T such that

$$\begin{cases} -\nu \Delta u + (u \cdot \nabla)u + \nabla p = \lambda j T, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ -\Delta T + \lambda u \cdot \nabla T = 0, & \text{in } \Omega, \\ u = 0, \quad T = T_0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^2$ be a bounded domain with boundary $\partial\Omega$, u denotes the fluid velocity vector field, p the pressure filed, T the temperature filed, $\nu > 0$ the coefficient of the kinematic viscosity, $\lambda > 0$ the Groshoff number, $j = (0, 1)$ the two-dimensional vector and T_0 the given initial scale function.

The stationary Conduction–Convection problem (I) is the coupled equations governing steady viscous incompressible flow and heat transfer process, where incompressible fluid is the Boussinesq approximation of the Navier–Stokes equations. In atmospheric dynamics it is an important compelling dissipative nonlinear system, which contains not only the velocity vector field and the pressure field but also the temperature field. From the thermodynamics point of view, we know that the movement of the fluid must have viscosity which will produce quantity of heat. Thus, the movement of the fluid must

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be companied with mutual transformation of temperature, speed and pressure. Therefore, it is very universal to investigate this nonlinear system.

For many engineering problems, the least squares formulation provides an attractive alternative to the standard Galerkin formulation. Irrespective of the type of the underlying partial differential equation, the least squares formulation always leads to symmetric system matrices, which implies that only half of the coefficients need to be stored. If, in addition, the system satisfies an a priori coercivity inequality, the least squares formulation generates positive definite algebraic system matrices, which allows one to use well-established solvers, such as preconditioned conjugate gradient methods. On the other hand, the least squares method does not need the conventional requirement of the inf-sup condition in mixed finite element methods. Recently, in an attempt to circumvent this constraint, the so-called CBB [5] or stabilized finite element methods [6–9] have been developed, motivated by SD (or SUPG) methods [10,11]. So far some studies have been done to the Conduction–Convection problem (cf. [1–4,12–14]), but these studies only given some numerical computational methods, much more less attention have been paid to the theoretical error analysis of the mixed finite element methods. In [15], a Galerkin least squares type finite element method is proposed and analyzed for the stationary Navier–Stokes equations. The existence, uniqueness and convergence of the discrete solution is proved in the case of sufficient viscosity. Afterwards [16] applied this method to the stationary Conduction–Convection problem, but all the analysis in [15,16] are about the conforming finite elements. Because of nonconforming elements have been used effectively especially in fluid and solid mechanics due to their stability. So we are concerned with nonconforming finite element methods for the stationary Conduction–Convection problem.

The main aim of this paper is to construct the least squares Galerkin–Petrov nonconforming mixed finite element method (LSGPNMFM) for the stationary Conduction–Convection problem. The existence, uniqueness and convergence of the discrete solution are presented and error estimates of optimal order are derived in the case of sufficient viscosity. In Section 2, we introduce the variational formulation for Problem (I) and the existence and uniqueness of variational formulation solution. In Section 3, we will give the construction of nonconforming mixed finite element scheme and present LSGPNMFM for the stationary Conduction–Convection problem. The existence and uniqueness of the LSGPNMFEM solution to the stationary Conduction–Convection problem will be proved in Section 4. In Section 5, the convergence analysis and optimal error estimates are obtained by using some important lemmas.

We will employ standard definitions for the Sobolev spaces $W^{k,p}(\Omega)$ with norm $\|\cdot\|_{k,p,\Omega}$, and $H^k(\Omega) = W^{k,2}(\Omega)$ with norm $\|\cdot\|_k$, respectively (cf. [17]). Throughout the paper, C indicates a positive constant, possibly different at different occurrences, which is independent of the mesh parameter h, but may depend on Ω and other parameters introduced in this paper. Notations not especially explained are used with their usual meanings.

2. The variational formulation

The variational formulation for Problem (I) is written as:

Problem (I^*) Find $(u, p, T) \in X \times M \times W$, such that $T|_{\partial\Omega} = T_0$, satisfying

$$\begin{cases} a(u, v) + a_1(u; u, v) - b(p, v) = \lambda(jT, v), & \forall\, v \in X, \\ b(q, u) = 0, & \forall\, q \in M, \\ d(T, \varphi) + \lambda \bar{a}_1(u; T, \varphi) = 0, & \forall\, \varphi \in W_0, \end{cases} \tag{2.1}$$

where

$$X = H_0^1(\Omega)^2, \qquad M = L_0^2(\Omega) = \left\{ q, q \in L^2(\Omega), \ \int_\Omega q \mathrm{d}x = 0 \right\}, \qquad W = H^1(\Omega), \qquad W_0 = H_0^1(\Omega),$$

$(\cdot, \cdot)$ means the inner product in $L^2(\Omega)^2$ or in $L^2(\Omega)$ according to the context, $a(u, v) = \nu(\nabla u, \nabla v)$, $b(p, v) = (p, \operatorname{div} v)$, $d(T, \varphi) = (\nabla T, \nabla \varphi)$,

$$a_1(u; v, w) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^{2} \Big(u^i \frac{\partial v^j}{\partial x_i} w^j - u^i \frac{\partial w^j}{\partial x_i} v^j \Big) \mathrm{d}x, \tag{2.2}$$

and

$$\bar{a}_1(u; T, \varphi) = \frac{1}{2} \int_\Omega \sum_{i=1}^{2} \Big(u^i \frac{\partial T}{\partial x_i} \varphi - u^i \frac{\partial \varphi}{\partial x_i} T \Big) \mathrm{d}x. \tag{2.3}$$

It has been shown in [18–21] that there exist positive constants $C_i > 0$ $(i = 1, 2, 3)$ and $C_j' > 0$ $(j = 1, 2)$ such that

$$\|v\|_0 \le C_1 \|\nabla v\|_0, \quad \forall\, v \in H_0^1(\Omega)^2, \qquad \|v\|_0 \le C_1' \|\nabla v\|_0, \quad \forall\, v \in H_0^1(\Omega), \tag{2.4}$$

$$\|v\|_{0,4} \le C_2 \|\nabla v\|_0, \quad \forall\, v \in H_0^1(\Omega)^2, \qquad \|v\|_{0,4} \le C_2' \|\nabla v\|_0, \quad \forall\, v \in H^1(\Omega), \tag{2.5}$$

$$\|v\|_{0,4} \le C_3 \|\nabla v\|_0, \quad \forall\, v \in H^1(\Omega)^2. \tag{2.6}$$