Nonlinear Analysis 72 (2010) 1726–1736

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Monodromy, center–focus and integrability problems for quasi-homogeneous polynomial systems

A. Algaba [a], E. Freire [b], E. Gamero [b,*], C. García [a]

[a] Department of Mathematics, Facultad de Ciencias, University of Huelva, Spain

[b] Department of Applied Mathematics II, E. S. I., University of Sevilla, Spain

ARTICLE INFO

Article history:
Received 17 April 2009
Accepted 1 September 2009

MSC:
34C05
34C07
37C25

Keywords:
Polynomial systems
Monodromy
Center–focus problem
Integrability

ABSTRACT

This paper deals with planar quasi-homogeneous polynomial vector fields, and addresses three major questions: the monodromy, the center–focus and the integrability problems. We characterize the monodromic planar quasi-homogeneous polynomial vector fields, and we give a condition to distinguish between a center and a focus in this case. Also, we provide conditions which characterize the integrability of quasi-homogeneous polynomial systems under non-resonance conditions. The results obtained allow us to analyse two monodromic planar systems with degenerate linear part: one of them with nilpotent linearization, and another one with null linear part.

1. Introduction

Nonlinear ordinary differential equations appear in many branches of applied sciences. In the context of planar systems, one of the basic questions is the monodromy problem, which concerns determining if the Poincaré first return map is defined for an equilibrium point. When this occurs the singular point is called monodromic and, in the analytic case, it is either a center or a focus.

Once the monodromy has been established, the classical Poincaré's center–focus problem looks for conditions that decide if a monodromic equilibrium is a center or a focus. In other words, one determines when all the orbits around an equilibrium point are closed, at least in a small neighborhood.

In such a case, the integrability problem arises: one determines if the planar vector field has a first integral, that is, a function which remains constant along the trajectories of the system. In a general framework, the integrability is an important question because the existence of a first integral completely determines its phase portrait.

It is well known that the presence of an analytic center assures the existence of local $\mathcal{C}^\infty$ first integrals (see Mazzi and Sabatini [1]). In general, it is not possible to assure the existence of local analytic first integrals, unless we restrict to a punctured neighborhood of the equilibrium (see Li et al. [2]).

In the nondegenerate case (that is, when the linear part of the equilibrium is equivalent to $(-y, x)^T$), the Poincaré Theorem states that there is a center in an analytic planar system if and only if there are analytic first integrals.

In the degenerate situation, there are two situations to be considered: the nilpotent and the zero linear part cases.

* Corresponding author.
E-mail address: estanis@esi.us.es (E. Gamero).

0362-546X/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2009.09.012

In the nilpotent case, there are analytic systems with centers which do not admit analytic first integrals (see Moussu [3]). The theoretical characterization of centers and analytic first integrals for analytic nilpotent systems can be found in Strózyna and Zoladek [4].

For the second case, corresponding to equilibria with zero linear part, only naive analyses have been carried out. The ideas presented in this paper should contribute to a deeper understanding of this case. In fact, some of our examples are devoted to this situation.

The subject of this paper is the analysis of the monodromy, center–focus and integrability problems for a concrete family of planar differential equations: those related to quasi-homogeneous (also called weighted-homogeneous) vector fields.

The quasi-homogeneity plays an important role in the study of the topological determination of a system (see Bruno [5], Brunella–Miari [6]). Namely, any vector field can be expanded in quasi-homogeneous terms of an arbitrary type. If a quasi-homogeneous term of the principal part is topologically determining for some specific type, then the full vector field is topologically equivalent to that term.

The quasi-homogeneity is also meaningful in the analysis of the monodromy problem (see Medvedeva [7]), as well as other areas of the dynamical systems theory. For instance, Algaba et al. [8] considered the normal form technique from a quasi-homogeneous perspective; Mañosa [9] gives conditions for a focus in degenerate monodromic systems having characteristic directions.

We can say that the use of quasi-homogeneity in the different techniques for the analysis of dynamical systems (blow-up, normal forms, Poincaré maps, etc.) is theoretically and computationally analogous to the homogeneous case, but the quasi-homogeneous framework is dynamically meaningful.

This paper is arranged in four sections. In the rest of the present section we establish some basic notations and give a way to split a quasi-homogeneous planar vector field as the sum of one with zero-divergence and another one with the divergence of the original vector field.

This splitting is used in Section 2 to describe the phase portraits of quasi-homogeneous planar systems. This is done in Proposition 2.6, where we characterize the quasi-homogeneous vector fields which have monodromy.

Later, in Section 3, assuming that we deal with a monodromic quasi-homogeneous system, we give in Theorem 3.7 a condition to distinguish between a center and a focus, which depends on the vanishing of a coefficient. Also, simple formulas to obtain the quoted coefficient are given in the simplest cases. The results in this section generalize those presented in Collins [10].

Finally, in Section 4 we provide conditions on a quasi-homogeneous vector field in order to have a formal or analytic first integral. To this end, we use again the splitting derived below. The main result is stated in Theorem 4.13, which characterizes the integrability of quasi-homogeneous systems under non-resonance conditions, providing also the structure of first integrals. Section 4 is completed with the analysis of two monodromic planar systems with degenerate linear part: one with nilpotent linearization, and other with null linear part.

1.1. Preliminary notations

Let us consider a planar system of the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}), \quad \text{with } \mathbf{x} = (x, y)^{\mathrm{T}} \in \mathbb{R}^2, \tag{1.1}$$

where $\mathbf{F} = (P, Q)^{\mathrm{T}}$ is a quasi-homogeneous (or weighted-homogeneous) vector field of type (or weigh) $\mathbf{t} = (t_1, t_2) \in \mathbb{N} \times \mathbb{N}$ and degree $r \in \mathbb{Z}$ ($\mathbb{N}$ is the set of natural numbers, not including zero; instead $\mathbb{Z}_+$ will denote the set of non-negative integers). Throughout this paper, we will assume that t_1, t_2 have no common factors (this can always be achieved by canceling them if they exist) and also that $t_1 \le t_2$ (otherwise, it is enough to interchange $x \leftrightarrow y$).

Recall that a scalar polynomial $p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $p(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k p(x, y)$ for all $\varepsilon \in \mathbb{R}$. The set of scalar quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$.

The vector field $\mathbf{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is called quasi-homogeneous of type $\mathbf{t}$ and degree r if its components P, Q are quasi-homogeneous polynomials of type $\mathbf{t}$ with degrees $r + t_1, r + t_2$ respectively. The set of quasi-homogeneous vector fields of type $\mathbf{t}$ and degree r will be denoted by $\mathcal{Q}_r^{\mathbf{t}}$.

There is a simple way to visualize quasi-homogeneous monomials, by drawing the *Newton diagram* (see Bruno [5], Dumortier [11]). As pointed out in Fig. 1, each point into the lattice $\mathbb{Z}_+ \times \mathbb{Z}_+$ is put in correspondence with quasi-homogeneous monomials, in both cases: scalar and vectorial. The points lying in the straightlines perpendicular to $\mathbf{t}$ determine the monomials having the same quasi-homogeneous degree.

There are a number of standard definitions that we will use in the following:

- If $\mathbf{t} = (t_1, t_2)$, then its modulus is defined as $|\mathbf{t}| = t_1 + t_2$.
- For a scalar function $f(x, y)$, we denote by $\mathbf{X}_f = \left(-\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)^{\mathrm{T}}$ the Hamiltonian vector field with Hamiltonian f.
- The divergence of a vector field $\mathbf{F} = (P, Q)^{\mathrm{T}}$ is $\mathrm{div}(\mathbf{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.
- The wedge product of two vector fields $\mathbf{F} = (P, Q)^{\mathrm{T}}$, $\mathbf{G} = (\tilde{P}, \tilde{Q})^{\mathrm{T}}$ is $\mathbf{F} \wedge \mathbf{G} = P\tilde{Q} - Q\tilde{P}$. Observe that $\mathbf{F} \wedge \mathbf{X}_f = \mathbf{F} \cdot \nabla f$.
- The Poisson bracket of two scalar functions p, q is $\{p, q\} = -\frac{\partial q}{\partial y}\frac{\partial p}{\partial x} + \frac{\partial q}{\partial x}\frac{\partial p}{\partial y}$. Notice that $\{p, q\} = \nabla q \wedge \nabla p = \mathbf{X}_q \wedge \mathbf{X}_p$.