Nonlinear Analysis 72 (2010) 1768–1777

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Smoothness and stability of the solutions for nonlinear fractional differential equations

Weihua Deng

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China

ARTICLE INFO

Article history:
Received 12 July 2008
Accepted 1 September 2009

MSC:
26A33
34A34
58B10
58K25

Keywords:
Fractional differential equation
Differentiability
Stability

ABSTRACT

This paper first obtains the differentiability properties of the solutions for nonlinear fractional differential equations, and then the sufficient conditions for the local asymptotical stability of nonlinear fractional differential equations are also derived.

1. Introduction

Fractional calculus is an area having a long history whose infancy dates back to three hundred years, the beginnings of classical calculus. It had always attracted the interest of many famous ancient mathematicians, including L'Hospital, Leibniz, Liouville, Riemann, Grünward, and Letnikov [1,2]. As the ancient mathematicians expected, in recent decades, fractional differential equations have been found to be a powerful tool in more and more fields, such as materials, physics, mechanics, and engineering [3–9,1,2,10].

Fractional differential operators are one kind of pseudo-differential operators. Since they are nonlocal and have weakly singular kernels, the study of fractional differential equations seems to be more difficult and less theories have been established than for classical differential equations. First, this paper studies the smoothness properties of the solutions for nonlinear fractional differential equations (FDEs). Not only the analytical properties of FDEs, but also, more important, the knowledge of the smoothness properties is indispensable for the construction of good numerical schemes [11–15]. This proves that, in general, the differentiability properties of the solutions for FDEs at initial points are different from other points. Thanks to these results, it is surprisingly found that it is hard for autonomous/nonautonomous FDEs to have periodic solutions besides fixed points, which are quite distinct from classical ODEs. On the other hand, it appears to be natural if one understands this phenomena from a physical point of view; the possible periodicity of the solutions for FDEs are destroyed by the *memory effects* of fractional differential operators [4]. It is well known that stability issue is a key topic for application sciences. The necessary and sufficient stability conditions for linear FDEs and linear time-delayed FDEs have already been obtained in [16,17,8]. To the best of our knowledge, the stability of nonlinear FDEs is still "open", although this is a very hot and urgent topic for engineers, physicists, controllers, and pure mathematicians. Many of the scientists are expecting the *fractional* version of the classical Hartman–Grobman theorem for hyperbolic dynamical systems of order 1 [5]. This paper provides the sufficient stability conditions for nonlinear FDEs.

E-mail addresses: dengwh@lzu.edu.cn, dengwhmath@yahoo.com.cn.

doi:10.1016/j.na.2009.09.018

The definition of fractional integral is described by

$$ {}_aD_t^{-\beta}x(t) = \frac{1}{\Gamma(\beta)}\int_a^t (t-\tau)^{\beta-1}x(\tau)\mathrm{d}\tau, \quad \beta > 0. $$

There are three kinds of widely used fractional derivatives, namely the Grünwald–Letnikov derivative, the Riemann–Liouville derivative, and the Caputo derivative. The Grünwald–Letnikov derivative and the Riemann–Liouville derivative are equivalent if the functions they act on are sufficiently smooth, and the Riemann–Liouville derivative is meaningful under weaker smoothness requirements. So just the definitions of the Riemann–Liouville derivative and the Caputo derivative are introduced as follows: the Riemann–Liouville derivative

$$ {}_aD_t^q x(t) = D^m {}_aD_t^{q-m}x(t), \quad q \in [m-1,\, m), $$

and the Caputo derivative

$$ {}_a^CD_t^q x(t) = {}_aD_t^{q-m}D^m x(t), \quad q \in (m-1,\, m), $$

where $m \in \mathbb{Z}^+$, D^m is the classical m-order derivative.

For the Riemann–Liouville derivative, we have

$$ \lim_{q\to(m-1)^+} {}_aD_t^q x(t) = \frac{\mathrm{d}^{m-1}x(t)}{\mathrm{d}t^{m-1}} $$

and

$$ \lim_{q\to m^-} {}_aD_t^q x(t) = \frac{\mathrm{d}^m x(t)}{\mathrm{d}t^m}. $$

But for the Caputo derivative, we have

$$ \lim_{q\to(m-1)^+} {}_a^CD_t^q x(t) = \frac{\mathrm{d}^{m-1}x(t)}{\mathrm{d}t^m} - D^{(m-1)}x(a) $$

and

$$ \lim_{q\to m^-} {}_a^CD_t^q x(t) = \frac{\mathrm{d}^m x(t)}{\mathrm{d}t^m}. $$

Obviously, ${}_aD_t^q$ $(q \in (-\infty,\, +\infty))$ varies continuously with q, i.e., ${}_aD_t^q$ $(q \in (-\infty,\, +\infty))$ bridges all the gaps among the integer derivatives and the integer integrals, but the Caputo derivative cannot do this [18]. However, the Caputo derivative is extensively used in real applications because the initial conditions of FDEs with Caputo derivative have a clear physical meaning [6,13].

This introduction is closed by outlining the rest of the paper. In the next section, some preliminary lemmas are first provided and then the smoothness of the solutions for nonlinear FDEs with the Caputo derivative and with the Riemann–Liouville derivative are discussed. The theorem on sufficient stability conditions of nonlinear FDEs with the Caputo derivative is proved in Section 3.

2. Smoothness of nonlinear FDEs

The FDE with the Caputo derivative is given as

$$ \begin{cases} {}_a^CD_t^q x(t) = f(t, x(t)), & m-1 < q < m \in Z^+, \\ D^k x(a) = x_a^k, & k = 0, 1, \ldots, m-1, \end{cases} \tag{1} $$

and the FDE with Riemann–Liouville derivative is provided by

$$ \begin{cases} {}_aD_t^q x(t) = f(t, x(t)), & m-1 < q < m \in Z^+, \\ \left[{}_aD_t^{q-k}x(t)\right]_{t=a} = x_a^k, & k = 1, 2, \ldots, m. \end{cases} \tag{2} $$

2.1. Preliminary lemmas

The FDEs (1) and (2) can be converted to their equivalent Volterra integral equations of the second kind under some natural conditions.

Lemma 2.1 ([19]). *If the function $f(t, x)$ is continuous, then the initial value problem (1) is equivalent to the following nonlinear Volterra integral equation of the second kind,*

$$ x(t) = \sum_{k=0}^{m-1} \frac{x_a^k}{k!}(t-a)^k + \frac{1}{\Gamma(q)}\int_a^t (t-\tau)^{q-1} f(\tau, x(\tau))\mathrm{d}\tau, \tag{3} $$

and its solutions are continuous.