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A linear wave equation with a nonlinear boundary condition of viscoelastic type

Le Thi Phuong Ngoc [a], Nguyen Anh Triet [b], Nguyen Thanh Long [c,*]

[a] Nhatrang Educational College, 01 Nguyen Chanh Street, Nhatrang City, Viet Nam

[b] Department of Mathematics, University of Architecture of Ho Chi Minh City, 196 Pasteur Street, Dist. 3, Ho Chi Minh City, Viet Nam

[c] Department of Mathematics and Computer Science, University of Natural Science, Vietnam National University, Ho Chi Minh City, 227 Nguyen Van Cu Street, Dist. 5, Ho Chi Minh City, Viet Nam

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ABSTRACT

The paper is devoted to the study of a linear wave equation with a nonlinear boundary condition of viscoelastic type. The existence of a weak solution is proved by using the Faedo–Galerkin method. The uniqueness, regularity and asymptotic expansion of the solution are also discussed.

1. Introduction

In this paper, we consider the initial-boundary value problem for the linear wave equation

$$u_{tt} - \frac{\partial}{\partial x}(\mu(x,t)u_x) + Ku + \lambda u_t = F(x,t), \quad 0 < x < 1,\ 0 < t < T, \tag{1.1}$$

$$\mu(0,t)u_x(0,t) = P_0(t), \tag{1.2}$$

$$-\mu(1,t)u_x(1,t) = P_1(t), \tag{1.3}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \tag{1.4}$$

$$P_i(t) = g_i(t) + K_i\,|u(i,t)|^{p_i-2}\,u(i,t) + \lambda_i\,|u_t(i,t)|^{q_i-2}\,u_t(i,t) - \int_0^t k_i(t-s)\,|u(i,s)|^{r_i-2}\,u(i,s)ds, \quad i = 0, 1, \tag{1.5}$$

where $K, \lambda \geq 0$, $p_i > 1$, $q_i \geq r_i > 1$, $K_i \geq 0$, $\lambda_i > 0$ are given constants and $u_0, u_1, \mu, F, g_i, k_i$ are given functions satisfying conditions specified later.

* Corresponding author.
E-mail addresses: phuongngoccdsp@dng.vnn.vn, ngoc1966@gmail.com (L.T.P. Ngoc), nguyenanhtriet@gmail.com (N.A. Triet), longnt@hcmc.netnam.vn, longnt2@gmail.com (N.T. Long).

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Various forms of the above problem have been studied by many authors; for example, we refer the reader to [1–7] and references therein.

In [1], An and Trieu studied a special case of the problem (1.1), (1.2), (1.4) and $(1.5)_{i=0}$, associated with the following homogeneous boundary condition at $x = 1$:

$$u(1, t) = 0, \tag{1.6}$$

with $\mu \equiv 1, F = u_0 = u_1 = 0, \lambda_0 = 0, p_0 = r_0 = 2$. The special case is a mathematical model describing the shock of a rigid body and a linear viscoelastic bar resting on a rigid base.

For the same problem with a different form, in [2], Bergounioux, Long and Dinh also studied Eqs. (1.1)–(1.5) corresponding to $\mu \equiv 1, g_1(t) = k_1(t) \equiv 0$, where $p_1 = q_1 = p_0 = r_0 = 2, \lambda_0 = 0$. This case is a mathematical model describing a shock problem involving a linear viscoelastic bar.

In [5], Munoz-Rivera and Andrade dealt with the global existence and exponential decay of solutions of the nonlinear one-dimensional wave equation with a viscoelastic boundary condition. The model studied there was given by

$$u_{tt}(x, t) - \frac{\partial}{\partial x}\left[\sigma\left(u_x(x, t)\right)\right] = f(x, t), \quad 0 < x < 1, t > 0, \tag{1.7}$$

$$u(0, t) = 0, \quad t > 0, \tag{1.8}$$

$$u(1, t) + \int_0^t a(t-s)\sigma(u_x(1, s))\mathrm{d}s = g(t), \quad t > 0, \tag{1.9}$$

and (1.4). Applying Volterra's inverse operator, (1.9) was transformed into

$$\sigma(u_x(1, t)) = G(t) - \frac{1}{a(0)}\left[u_t(1, t) + k(0)u(1, t) + \int_0^t k'(t-s)u(1, s)\,\mathrm{d}s\right], \quad t > 0, \tag{1.10}$$

where $G(t) = \frac{1}{a(0)}[k(t)\,u_0(1) + g'(t) + \int_0^t k(t-s)\,g'(s)\mathrm{d}s]$, and the resolvent kernel satisfies

$$k(t) + \frac{1}{a(0)}\int_0^t a'(t-s)k(s)\,\mathrm{d}s = \frac{-1}{a(0)}a'(t). \tag{1.11}$$

In [7], Santos studied the asymptotic behavior of the solution of the problem (1.1), (1.4) and (1.8) with $\mu = \mu(t), F = 0$ associated with a boundary condition of memory type at $x = 1$ as follows:

$$u(1, t) + \int_0^t a(t-s)\mu(s)u_x(1, s)\mathrm{d}s = 0, \quad t > 0. \tag{1.12}$$

Santos also transformed (1.12) into

$$-\mu(t)u_x(1, t) = g_1(t) + K_1u(1, t) + \lambda_1u_t(1, t) - \int_0^t k_1(t-s)u(1, s)\mathrm{d}s, \tag{1.13}$$

where $K_1 = \frac{k(0)}{a(0)}, \lambda_1 = \frac{1}{a(0)}$ are positive constants and $k_1(t) = \frac{-1}{a(0)}k'(t), g_1(t) = -\frac{1}{a(0)}u_0(1)k(t)$.

On the basis of the above works, we will study in this paper the existence result as well as the uniqueness and asymptotic expansion of the solution to the problem (1.1)–(1.5), with the nonlinear boundary condition of viscoelastic type at $x = 0, x = 1$. We consider three main parts as follows.

In Part 1, for the conditions $(u_0, u_1) \in H^1 \times L^2, F \in L^1(0, T; L^2), k_i \in L^1(0, T), g_i \in L^{q_i'}(0, T), \mu \in C^0\left(\overline{Q_T}\right), \mu(x, t) \geq \mu_0 > 0, \mu_t \in L^1(0, T; L^\infty), \mu_t(x, t) \leq 0$, a.e. $(x, t) \in Q_T, \lambda_i > 0; p_i > 1, q_i \geq r_i > 1, q_i' = \frac{q_i}{q_i - 1}, (K, \lambda, K_0, K_1) \in \mathbb{R} \times \mathbb{R}_+^3$, we prove that the problem (1.1)–(1.5) has a weak solution u. If, in addition, $k_i \in W^{1,1}(0, T)$ and $p_i \in \{2\} \cup [3, +\infty), r_i \geq 2$, then u is unique. The proof is based on the Faedo–Galerkin method and the weak compact method associated with a monotone operator.

In Part 2, with $p_i \geq q_i = 2, r_i > 1, p_i \geq \max\{2, 2r_i - 2\}$, we establish conditions in order to imply that u belongs to $L^\infty(0, T; H^2)$, with $u_t \in L^\infty(0, T; H^1), u_{tt} \in L^\infty(0, T; L^2), u(0, \cdot), u(1, \cdot) \in H^2(0, T)$.

Finally, in Part 3, with $\lambda_i = 1; q_i = r_i = 2; p_i \geq N + 1, N \geq 2$, we give an asymptotic expansion of the solution u of the problem (1.1)–(1.5) up to order $N + 1$ in four small parameters K, λ, K_0, K_1. The results obtained here may be considered as generalizations of those in [1–12].

2. The existence and uniqueness theorem

Put $\Omega = (0, 1), Q_T = \Omega \times (0, T), T > 0$. We omit the definitions of the usual function spaces: $C^m\left(\overline{\Omega}\right), L^p(\Omega), W^{m,p}(\Omega)$. We define $W^{m,p} = W^{m,p}(\Omega), L^p = W^{0,p}(\Omega), H^m = W^{m,2}(\Omega), 1 \leq p \leq \infty, m = 0, 1, \ldots$. The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2 or a pair of dual scalar products of continuous linear functionals with an element of a function space. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X. We write $L^p(0, T; X), 1 \leq p \leq \infty$, for the Banach space of the real functions $u : (0, T) \to X$, measurable, such that