Nonlinear Analysis 72 (2010) 1916–1921

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Implicit function theorem via the DSM

A.G. Ramm

Department of Mathematics, Kansas State University, Manhattan, KS 66506-2602, USA

ARTICLE INFO

Article history:
Received 29 July 2009
Accepted 14 September 2009

MSC:
47J05
47J25

Keywords:
Dynamical systems method (DSM)
Hard implicit function theorem
Newton's method

ABSTRACT

Sufficient conditions are given for an implicit function theorem to hold. The result is established by an application of the Dynamical Systems Method (DSM). It allows one to solve a class of nonlinear operator equations in the case when the Fréchet derivative of the nonlinear operator is a smoothing operator, so that its inverse is an unbounded operator.

1. Introduction

The aim of this paper is to demonstrate the power of the Dynamical Systems Method (DSM) as a tool for proving theoretical results. The DSM was systematically developed in [1] and applied to solving nonlinear operator equations in [1] (see also [2]), where the emphasis was on convergence and stability of the DSM-based algorithms for solving operator equations, especially nonlinear and ill-posed equations. The DSM for solving an operator equation $F(u) = h$ consists of finding a nonlinear map $u \mapsto \Phi(t, u)$, depending on a parameter $t \in [0, \infty)$, that has the following three properties:

(1) the Cauchy problem

$$\dot{u} = \Phi(t, u), \qquad u(0) = u_0 \quad \left(\dot{u} := \frac{\mathrm{d}u(t)}{\mathrm{d}t}\right)$$

has a unique global solution $u(t)$ for a given initial approximation u_0;

(2) the limit $u(\infty) = \lim_{t\to\infty} u(t)$ exists; and

(3) this limit solves the original equation $F(u) = h$, i.e., $F(u(\infty)) = h$.

The operator $F : H \to H$ is a nonlinear map in a Hilbert space H. It is assumed that the equation $F(u) = h$ has a solution, possibly nonunique.

The problem is to find a Φ such that the properties (1), (2), and (3) hold. Various choices of Φ for which these properties hold are proposed in [1], where the DSM is justified for wide classes of operator equations, in particular, for some classes of nonlinear ill-posed equations (i.e., equations $F(u) = 0$ for which the linear operator $F'(u)$ is not boundedly invertible). By $F'(u)$ we denote the Fréchet derivative of the nonlinear map F at the element u.

In this note the DSM is used as a tool for proving a "hard" implicit function theorem.

Let us first recall the usual implicit function theorem. Let U solve the equation $F(U) = f$.

E-mail address: ramm@math.ksu.edu.

0362-546X/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2009.09.032

Proposition. *If $F(U) = f$, F is a C^1-map in a Hilbert space H, and $F'(U)$ is a boundedly invertible operator, i.e., $\|[F'(U)]^{-1}\| \le m$, then the equation*

$$F(u) = h \tag{1.1}$$

is uniquely solvable for every h sufficiently close to f.

For convenience of the reader we include a proof of this known result.

Proof of the Proposition. First, one can reduce the problem to the case $u = 0$ and $h = 0$. This is done as follows. Let $u = U + z$, $h - f = p$, $F(U + z) - F(U) := \phi(z)$. Then $\phi(0) = 0$, $\phi'(0) = F'(U)$, and Eq. (1.1) is equivalent to the equation

$$\phi(z) = p, \tag{1.2}$$

with the assumptions

$$\phi(0) = 0, \qquad \lim_{z\to 0} \|\phi'(z) - \phi'(0)\| = 0, \qquad \|[\phi'(0)]^{-1}\| \le m. \tag{1.3}$$

We want to prove that Eq. (1.2) under the assumptions (1.3) has a unique solution $z = z(p)$, such that $z(0) = 0$, and $\lim_{p\to 0} z(p) = 0$. To prove this, consider the equation

$$z = z - [\phi'(0)]^{-1}(\phi(z) - p) := B(z), \tag{1.4}$$

and check that the operator B is a contraction in a ball $\mathcal{B}_\epsilon := \{z : \|z\| \le \epsilon\}$ if $\epsilon > 0$ is sufficiently small, and B maps $\mathcal{B}_\epsilon$ into itself. If this is proved, then the desired result follows from the contraction mapping principle.

One has

$$\|B(z)\| = \|z - [\phi'(0)]^{-1}(\phi'(0)z + \eta - p)\| \le m\|\eta\| + m\|p\|, \tag{1.5}$$

where $\|\eta\| = o(\|z\|)$. If ϵ is so small that $m\|\eta\| < \frac{\epsilon}{2}$ and p is so small that $m\|p\| < \frac{\epsilon}{2}$, then $\|B(z)\| < \epsilon$, so $B : \mathcal{B}_\epsilon \to \mathcal{B}_\epsilon$.

Let us check that B is a contraction mapping in $\mathcal{B}_\epsilon$. One has:

$$\begin{aligned}\|Bz - By\| &= \|z - y - [\phi'(0)]^{-1}(\phi(z) - \phi(y))\| \\ &= \|z - y - [\phi'(0)]^{-1} \int_0^1 \phi'(y + t(z - y))\mathrm{d}t(z - y)\| \\ &\le m \int_0^1 \|\phi'(y + t(z - y)) - \phi'(0)\|\mathrm{d}t\|z - y\|.\end{aligned} \tag{1.6}$$

If $y, z \in \mathcal{B}_\epsilon$, then

$$\sup_{0\le t\le 1} \|\phi'(y + t(z - y)) - \phi'(0)\| = o(1), \quad \epsilon \to 0.$$

Therefore, if ϵ is so small that $mo(1) < 1$, then B is a contraction mapping in $\mathcal{B}_\epsilon$, and Eq. (1.2) has a unique solution $z = z(p)$ in $\mathcal{B}_\epsilon$, such that $z(0) = 0$. The proof is complete. □

The crucial assumptions, on which this proof is based, are assumptions (1.3).

Suppose now that $\phi'(0)$ *is not boundedly invertible, so that the last assumption in* (1.3) *is not valid*. Then a theorem which still guarantees the existence of a solution to Eq. (1.2) for some set of p is called a "hard" implicit function theorem. Examples of such theorems one may find, e.g., in [3–6].

Our goal in this paper is to establish a new theorem of this type using a new method of proof, based on the Dynamical Systems Method (DSM). In [7] we have demonstrated a theoretical application of the DSM by establishing some surjectivity results for nonlinear operators.

The result, presented in this paper, is a new illustration of the applicability of the DSM as a tool for proving theoretical results.

To formulate the result, let us introduce the notion of a scale of Hilbert spaces H_a (see [8]). Let $H_a \subset H_b$ and $\|u\|_b \le \|u\|_a$ if $a \ge b$. Example of spaces H_a is the scale of Sobolev spaces $H_a = W^{a,2}(D)$, where $D \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary.

Consider Eq. (1.1). Assume that

$$F(U) = f; \qquad F : H_a \to H_{a+\delta}, \qquad u \in B(U, R) := B_a(U, R), \tag{1.7}$$

where $B_a(U, R) := \{u : \|u - U\|_a \le R\}$ and $\delta = \text{const} > 0$, and the operator $F : H_a \to H_{a+\delta}$ is continuous. Furthermore, assume that $A := A(u) := F'(u)$ exists and is an isomorphism of H_a onto $H_{a+\delta}$:

$$c_0\|v\|_a \le \|A(u)v\|_{a+\delta} \le c_0'\|v\|_a, \quad u, v \in B(U, R), \tag{1.8}$$