Nonlinear Analysis 72 (2010) 2100–2113

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Instability of bound states of nonlinear Schrödinger equations with Morse index equal to two

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ARTICLE INFO

Article history:
Received 25 March 2009
Accepted 9 October 2009

Keywords:
Nonlinear Schrödinger equation
Bound states
Orbital stability

ABSTRACT

We prove that every bound state of the nonlinear Schrödinger equation (NLS) with Morse index equal to two, with $\frac{d^2}{d\omega^2}(E(\phi_\omega)+\omega Q(\phi_\omega)) > 0$, is orbitally unstable. We apply this result to two particular cases. One is the NLS equation with potential and the other is a system of three coupled NLS equations. In both the cases the linear instability is well known but the orbital instability results are new when the spatial dimension is high.

1. Introduction

In this paper, we consider the instability of bound states of the nonlinear Schrödinger equation:

$$\mathrm{i}\partial_t u = -\Delta u + f(x,|u|^2)u, \quad (x,t)\in\mathbb{R}^{N+1}, \tag{1}$$

where $f(x,s) \in C^1(\mathbb{R}^N\times\mathbb{R}_+;\mathbb{R})$ and $\int_{\mathbb{R}^N}|F(x,|u|^2)|\,\mathrm{d}x < \infty$ for every $u\in H^1(\mathbb{R}^N)$. Here, we put $\mathbb{R}_+ = [0,\infty)$ and $F(x,s)=\int_0^s f(x,\tau)\,\mathrm{d}\tau$.

The nonlinear Schrödinger equation (1) arises in various physical contexts such as Bose–Einstein condensation, plasma physics and nonlinear optics (see e.g., [11,12,15]). By a bound state, we mean a nontrivial solution of (1) of the form $u(x,t)=\mathrm{e}^{\mathrm{i}\omega t}\phi_\omega(x)$, where $\omega\in\mathbb{R}$ and ϕ_ω is the solution of

$$-\Delta\phi_\omega+\omega\phi_\omega+f(x,|\phi_\omega|^2)\phi_\omega=0. \tag{2}$$

Since one can only observe the stable bound states in physical phenomena, it is important to investigate the stability of the bound states.

We define the energy E and the charge Q on $H^1(\mathbb{R}^N)$ by

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}F(x,|u|^2)\,\mathrm{d}x,$$
$$Q(u) := \frac{1}{2}\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x.$$

These quantities are formally conserved by the flow of (1).

Now, let $\mathcal{O}(N)$ be the orthogonal group in $\mathbb{R}^N$. Let $G\subset\mathcal{O}(N)$ be a subgroup, such that the nonlinearity $f(x,s)$ of (1) is invariant under G, i.e., $f(gx,s)=f(x,s)$ for every $g\in G$, $x\in\mathbb{R}^N$ and $s\geq 0$. We define a closed subspace $H^1_G(\mathbb{R}^N)$ of $H^1(\mathbb{R}^N)$ by

$$H^1_G(\mathbb{R}^N) := \left\{u\in H^1(\mathbb{R}^N) \mid u(gx)=u(x),\ g\in G,\ x\in\mathbb{R}^N\right\}. \tag{3}$$

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doi:10.1016/j.na.2009.10.010

We note that $H^1_G(\mathbb{R}^N) = H^1(\mathbb{R}^N)$ if $G = \{\mathrm{Id}$ (identity matrix)$\}$, and $H^1_G(\mathbb{R}^N) = H^1_r(\mathbb{R}^N)$ if $G = \mathcal{O}(N)$, where $H^1_r(\mathbb{R}^N) = \left\{u \in H^1(\mathbb{R}^N) \mid u(x) = u(|x|),\ x \in \mathbb{R}^N\right\}$. Let $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ be defined as

$$(u, v) := \mathrm{Re}\int_{\mathbb{R}^N} \nabla u \cdot \overline{\nabla v}\,\mathrm{d}x + \mathrm{Re}\int_{\mathbb{R}^N} u\overline{v}\,\mathrm{d}x,$$

$$\langle u, v\rangle := \mathrm{Re}\int_{\mathbb{R}^N} u\overline{v}\,\mathrm{d}x.$$

We regard $H^1_G(\mathbb{R}^N)$ as a real Hilbert space equipped with the inner product $(\cdot,\cdot)$. We further define the norms $\|\cdot\|_{H^1}$, $\|\cdot\|_{L^2}$ as

$$\|u\|^2_{H^1} := (u, u), \qquad \|u\|^2_{L^2} := \langle u, u\rangle .$$

We assume the local well-posedness of the Cauchy problem for (1) in $H^1_G(\mathbb{R}^N)$, and that the energy and the charge are conserved during the interval of existence.

Assumption 1 (*Existence of Solutions*). For every $\mu > 0$, there exists $t_0 > 0$ such that for every $u_0 \in H^1_G(\mathbb{R}^N)$ with $\|u_0\|_{H^1} \le \mu$, there exists a solution $u \in C([0, t_0); H^1_G(\mathbb{R}^N))$ of (1) such that

(a) $u(x, 0) = u_0(x)$ for $x \in \mathbb{R}^N$,
(b) $E(u(t)) = E(u_0)$, $Q(u(t)) = Q(u_0)$ for $t \in \mathcal{I} = [0, t_0)$.

We also assume the existence of bound states.

Assumption 2 (*Existence of Bound States*). There exist $\omega_1, \omega_2 \in \mathbb{R}$ with $\omega_1 < \omega_2$ and a C^1 mapping $(\omega_1, \omega_2) \ni \omega \mapsto \phi_\omega \in H^1_G(\mathbb{R}^N) \setminus \{0\}$ such that ϕ_ω satisfies (2). Further, $f(x, |\phi_\omega|^2)$, $\partial_s f(x, |\phi_\omega|^2)|\phi_\omega|^2 \in L^\infty$.

Definition 1 (*Orbital Stability*). We say that the bound state $\mathrm{e}^{\mathrm{i}\omega t}\phi_\omega$ is orbitally stable in $H^1_G(\mathbb{R}^N)$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u_0 \in H^1_G(\mathbb{R}^N)$ with $\|u_0 - \phi_\omega\|_{H^1} < \delta$, the solution u of (1) with the initial data $u(0) = u_0$ exists globally in time and satisfies

$$\sup_{t\in(0,\infty)} \inf_{s\in\mathbb{R}} \|u(t) - \mathrm{e}^{\mathrm{i}s}\phi_\omega\|_{H^1} < \varepsilon. \tag{4}$$

Otherwise, we say the bound state $\mathrm{e}^{\mathrm{i}\omega t}\phi_\omega$ is orbitally unstable in $H^1_G(\mathbb{R}^N)$.

The stationary problem (2) has a variational structure. That is, $\phi_\omega \in H^1(\mathbb{R}^N)$ is a solution of (2) if and only if ϕ_ω is a critical point of the action S_ω, where

$$S_\omega(u) := E(u) + \omega Q(u).$$

The Morse index of a critical point ϕ_ω of S_ω is the space dimension of the space which is spanned by the eigenvectors of the negative eigenvalues of $S''_\omega(\phi_\omega)$, where $S''_\omega(\phi_\omega)$ is the second Fréchet derivative at ϕ_ω of S_ω. Here, by a direct calculation, we have

$$S''_\omega(\phi_\omega)u = -\Delta u + \omega u + f(x, |\phi_\omega|^2)u + 2\partial_s f(x, |\phi_\omega|^2)\mathrm{Re}(\phi_\omega\overline{u})\phi_\omega. \tag{5}$$

We note that $S''_\omega(\phi_\omega)$ is a self-adjoint operator on the real Hilbert space $L^2(\mathbb{R}^N)$.

The ground states are the bound states which minimize S_ω among all the bound states. The excited states are the bound states which are not the ground states. In many situations, the Morse index of the ground states is 1 and the Morse index of the excited states is more than 1. There are many results for the stability and instability of the ground states of the nonlinear Schrödinger equation (1). When $f(x, s) = -s^\alpha$, for $0 < \alpha < 2/N$, the ground states are stable [2] and for $2/N \le \alpha < \alpha(N)$, the ground states are unstable [1,16]. Here, we put $\alpha(N) = \infty$ for $N = 1, 2$ and $\alpha(N) = 2/(N-2)$ for $N \ge 3$. In more general setting, for the case when Morse index is 1, Grillakis, Shatah and Strauss [8] have proved by an abstract theory that if $\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}(S_\omega(\phi_\omega)) > 0$ (resp. < 0), then $\mathrm{e}^{\mathrm{i}\omega t}\phi_\omega$ is orbitally stable (resp. unstable). For the proof of the instability, they use the fact that on the hypersurface $\left\{u \in H^1_G \mid Q(u) = Q(\phi_\omega)\right\}$, there is exactly one direction in which the energy decreases.

We now introduce another kind of instability. Let $\mathrm{e}^{\mathrm{i}\omega t}(\phi_\omega + \rho)$ be the solution of (1). Then ρ satisfies

$$\partial_t\rho = \mathcal{L}\rho + g_{\phi_\omega}(x, \rho),$$

where

$$\mathcal{L}\rho = -\mathrm{i}S''_\omega(\phi_\omega)\rho,$$

$$g_{\phi_\omega}(x, \rho) = -\mathrm{i}f(x, |\phi_\omega + \rho|^2)(\phi_\omega + \rho) + \mathrm{i}f(x, |\phi_\omega|^2)\phi_\omega + \mathrm{i}f(x, |\phi_\omega|^2)\rho + 2\mathrm{i}\partial_s f(x, |\phi_\omega|^2)\mathrm{Re}(\phi_\omega\overline{\rho})\phi_\omega.$$

We say the bound state $\mathrm{e}^{\mathrm{i}\omega t}\phi_\omega$ is linearly unstable if the linearized operator $\mathcal{L}$ has an eigenvalue with positive real part. In [9], it is shown that if the Morse index is even (resp. odd) and $\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}(S_\omega(\phi_\omega)) > 0$ (resp. < 0), then the bound state $\mathrm{e}^{\mathrm{i}\omega t}\phi_\omega$ is linearly unstable. In addition, it is shown that the orbital instability follows from the linear instability and the estimate

$$\|g_{\phi_\omega}(x, \rho)\|_{H^1} \le C\|\rho\|^{1+\alpha}_{H^1}, \quad \alpha > 0. \tag{6}$$