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Oscillation of damped PDE with p-Laplacian in unbounded domains

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To the memory of my parents

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ABSTRACT

In this paper, new oscillation theorems for the damped partial differential equation (PDE) with p-Laplacian

$$\operatorname{div}(A(x)\|\nabla u\|^{p-2}\nabla u) + \langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \rangle + c(x)|u|^{p-2}u = 0$$

are established in unbounded domains. Our theorems as two special cases when $A \equiv I$ or $A \equiv I$, $\vec{b}(x) \equiv 0$ improve and complement some existing results in the literature. To illustrate our main results, we give some corollaries and examples.

1. Introduction

Consider the damped partial differential equation (PDE) with p-Laplacian

$$\operatorname{div}(A(x)\|\nabla u\|^{p-2}\nabla u) + \langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u \rangle + c(x)|u|^{p-2}u = 0, \tag{1.1}$$

where $p > 1, x = (x_i)_{i=1}^N \in \Omega(1) := \{x \in \mathbb{R}^N : \|x\| \geq 1\} \subset \mathbb{R}^N$, $\|\cdot\|$ is the usual Euclidean norm in $\mathbb{R}^N$ and $\nabla = (\partial/\partial x_i)_{i=1}^N$ is the usual nabla operator. $A(x) = (a_{ij}(x))$ is an elliptic $N \times N$ matrix with differentiable components, $c(x)$ and $\vec{b}(x) = (b_i(x))_{i=1}^N$ are Hölder continuous function in $\Omega(1)$.

A solution of Eq. (1.1) in $\Omega(1)$ we understand a differentiable function $u = u(x)$ such that $A(x)\|\nabla u\|^{p-2}\nabla u$ is also differentiable and u satisfies Eq. (1.1) in $\Omega(1)$. Regarding the question of existence of the solutions of Eq. (1.1), we refer the reader to the monograph [1]. Eq. (1.1) is important for applications in physics, biology, glaciology, etc.;see [1,2].

Eq. (1.1) includes the following equations extensively studied in the literature.

(1) The linear Schrödinger partial differential equation

$$\Delta u + c(x)u = 0. \tag{1.2}$$

(2) The undamped PDE with p-Laplacian

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u) + c(x)|u|^{p-2}u = 0. \tag{1.3}$$

(3) The damped PDE with p-Laplacian

$$\operatorname{div}(\|\nabla u\|^{p-2}\nabla u) + \langle \vec{b}(x), \|\nabla u\|^{p-2}\nabla u\rangle + c(x)|u|^{p-2}u = 0. \tag{1.4}$$

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The oscillation theory of Eq. (1.2) deals with two types of oscillation. According to this theory, Eq. (1.2) is said to be *weakly oscillatory* if every solution has a zero outside of every ball in $\mathbb{R}^N$ and *strongly oscillatory* if every solution has a nodal domain outside of every ball in $\mathbb{R}^N$. The equivalence between these two types of oscillation for Eq. (1.2) has been proved in [3] for the local Hölder continuous function $c(x)$, which is a usual assumption concerning the smoothness of the function $c(x)$; see also [4] for short discussion concerning the general situation $p \neq 2$. In the connection to Eq. (1.1) we will use the following concepts of oscillation; see [2,5,6].

Definition 1.1. The function u defined in $\Omega(1)$ is said to be *oscillatory*, if the set of zeros of the function u is unbounded with respect to the norm. Eq. (1.1) is said to be *oscillatory* if every solution defined in $\Omega(1)$ is oscillatory.

Definition 1.2. Let Ω be an unbounded domain in $\mathbb{R}^N$. The function u defined in $\Omega(1)$ is said to be *oscillatory in* Ω, if the set of zeros of the function u, which lies in the closure $\overline{\Omega}$, is unbounded with respect to the norm. Eq. (1.1) is said to be oscillatory in Ω if every solution defined on $\Omega(1)$ is oscillatory in Ω. The equation is said to be nonoscillatory (nonoscillatory in Ω) if it is not oscillatory (oscillatory in Ω).

Since the pioneering work of Noussair and Swanson [7], there have been extensive investigations on oscillation (by Definition 1.1) for Eqs. (1.1)–(1.4); see, for example [2,4,8–17]. However, to the best of our knowledge, very little is known about the oscillation of Eqs. (1.1)–(1.4) on another type of unbounded domain, than an exterior of a ball. Here, we would like to mention the recent work of Mařík [5], in which Philos-type oscillation criteria [18] for different type of unbounded domains were derived for Eq. (1.3). Furthermore, in the paper of Mařík [6], Wintner-type oscillation theorem [19] as well as Philos-type oscillation criteria are established for Eq. (1.4) in unbounded domains. However, we have found that the Philos-type oscillation theorem has not been well developed for Eqs. (1.3) and (1.4) in [5,6]; see Remark 4.1.

The purpose of this paper is to establish new oscillation theorems for Eq. (1.1) in unbounded domains. By using the Riccati technique [7] and the modified integral averaging technique [20–22], we try to extend the results in [18,19] to Eq. (1.1), which improve and complement the main results in [5,6]. The criteria can detect also the oscillation over the more general exterior domains, then the exterior of some ball. The main feature in our results is that the oscillation criteria are not radially symmetric and do not depend on the mean value of the coefficients. To illustrate our results, we give a series of corollaries and examples.

2. Preliminaries

For simplicity, let

$$\Omega(a) = \{x \in \mathbb{R}^N : \|x\| \geq a\},$$
$$\Omega(a, b) = \{x \in \mathbb{R}^N : a \leq \|x\| \leq b\},$$
$$S(a) = \{x \in \mathbb{R}^N : \|x\| = a\}.$$

Let Ω be an unbounded domain in $\mathbb{R}^N$ and $a_0 \geq 1$. Define

$$D = \{(t, x) \in \mathbb{R} \times \overline{\Omega} \cap \Omega(a_0) : a_0 \leq \|x\| \leq t\},$$
$$D_0 = \{(t, x) \in \mathbb{R} \times \overline{\Omega} \cap \Omega(a_0) : a_0 \leq \|x\| < t\}.$$

Let the function $H \in \mathbf{C}(D, \mathbb{R}_0^+) \cap \mathbf{C}^1(D_0, \mathbb{R}_0^+)$ satisfy the following conditions.

(H1) $H(t, x) \equiv 0$ for $x \notin \overline{\Omega}$;

(H2) if $x \in \Omega^0$, then $H(t, x) = 0$ if and only if $t = \|x\|$;

(H3) there exists a function $\kappa \in \mathbf{C}([a_0, \infty), \mathbb{R}^+)$ such that

$$\mathcal{K}(t, r) := \kappa(r) \int_{\overline{\Omega} \cap S(r)} H(t, x) \mathrm{d}\sigma$$

is positive and has a nonpositive continuous partial derivative $\partial \mathcal{K}(t, r)/\partial r$ on D_0 for every fixed $t > r$;

(H4) it holds

$$0 < \inf_{s \geq a_0} \left[\liminf_{r \to \infty} \frac{\mathcal{K}(r, s)}{\mathcal{K}(r, a_0)} \right] \leq \infty,$$

where $\overline{\Omega}$ denotes the closure of Ω, $\mathrm{d}\sigma$ is the element of the surface of the sphere $S(r)$.

For later use, we introduce some notations. For $a_0 \geq 1$, $\beta \geq 1$, $\rho \in \mathbf{C}^1(\Omega(a_0), \mathbb{R}^+)$ and $H \in \mathbf{C}(D, \mathbb{R}_0^+) \cap \mathbf{C}^1(D_0, \mathbb{R}_0^+)$. Define

$$l(x) = \rho(x) \lambda_{\min}(x) \lambda_{\max}^{-q}(x),$$
$$g(x) = \rho(x) \lambda_{\min}^{1-p}(x) \lambda_{\max}^{p}(x),$$