Nonlinear Analysis 72 (2010) 2426–2443

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Expanding the asymptotic explosive boundary behavior of large solutions to a semilinear elliptic equation

S. Alarcón [a], G. Díaz [b,*], R. Letelier [1], J.M. Rey [b]

[a] Dpto. de Matemática, U. Técnica Federico Santa María, Casilla 110-V Valparaíso, Chile

[b] Dpto. de Matemática Aplicada, U. Complutense de Madrid, 28040 Madrid, Spain

ARTICLE INFO

Article history:
Received 2 October 2009
Accepted 31 October 2009

To René Letelier, in memoriam

MSC:
35B40
35J25
35J65

Keywords:
Large solutions
Asymptotic behavior
Upper and lower solutions

ABSTRACT

The main goal of this paper is to study the asymptotic expansion near the boundary of the large solutions of the equation

$$-\Delta u + \lambda u^m = f \quad \textit{in } \Omega,$$

where $\lambda > 0, m > 1, f \in \mathcal{C}(\Omega), f \geq 0$, and Ω is an open bounded set of $\mathbb{R}^N$, $N > 1$, with boundary smooth enough. Roughly speaking, we show that the number of explosive terms in the asymptotic boundary expansion of the solution is finite, but it goes to infinity as m goes to 1. We prove that the expansion consists in two eventual geometrical and non-geometrical parts separated by a term independent on the geometry of $\partial\Omega$, but dependent on the diffusion. For low explosive sources the non-geometrical part does not exist; all coefficients depend on the diffusion and the geometry of the domain by means of well-known properties of the distance function $\mathrm{dist}(x, \partial\Omega)$. For high explosive sources the preliminary coefficients, relative to the non-geometrical part, are independent on Ω and the diffusion. Finally, the geometrical part does not exist for very high explosive sources.

1. Introduction

In this paper we are interested in the solutions of the equation

$$-\Delta u + g(u) = f \quad \text{in } \Omega, \tag{1}$$

with an *explosive* behavior on the boundary

$$u(x) \to \infty \quad \text{as } x \to \partial\Omega. \tag{2}$$

In general, the solutions of (1) and (2) are called *large solutions* if a Comparison Principle holds. This is because the inequality

$$u(x) \geq v(x), \quad x \in \overline{\Omega},$$

is satisfied for any other solution v of (1) with bounded boundary values.

Singular boundary value problems as (1)–(2) have been extensively studied in the literature starting with the results of L. Bieberbach and H. Rademacher for precise choices of the function g (see for instance [1–4]). From our point of view, the pioneer works in the topic are due to Keller [5] and Osserman [6] on 1957 who proved the existence of large solutions of (1)

* Corresponding author.
E-mail addresses: salomon.alarcon@usm.cl (S. Alarcón), gdiaz@mat.ucm.es (G. Díaz), jrey@mat.ucm.es (J.M. Rey).

1 Deceased author.

0362-546X/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2009.10.040

provided that $f\equiv 0$, g is a nondecreasing function and Ω is a bounded open set of $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N}>1$. They also establish necessary and sufficient conditions to guarantee that the large solutions exist under the so called *Keller–Osserman* condition

$$\int^{\infty}\frac{\mathrm{d}s}{\sqrt{\int_0^s g(\tau)\mathrm{d}\tau}}<+\infty. \tag{3}$$

From that time forward an extensive literature has been produced (see again [1–4,7] and the references therein). In sight of results in [3] or [7] about the existence and uniqueness of the classical large solutions of (1), we focus our attention on their asymptotic behavior on the boundary $\partial\Omega$.

As it is usual in studying properties near the boundary, the distance function $\mathrm{dist}(x,\partial\Omega)$, here denoted by $\mathrm{d}(x)$, plays an important role. As it is well known, if the boundary is bounded with $\partial\Omega\in\mathcal{C}^k$, $k\geq 1$, one proves $\mathrm{d}(\cdot)\in\mathcal{C}^k$ in the parallel strip near the boundary

$$\Omega_{\delta_0}=\{x\in\Omega:\ 0\leq \mathrm{d}(x)<\delta_0\}. \tag{4}$$

Obviously, the positive constant δ_0 only depends on $\partial\Omega$ (see [2] or [8]). In particular, as it was proved in [3] if $\partial\Omega\in\mathcal{C}^2$ then the first term of the boundary explosive expansion is uniform and independent on Ω for the large solution of

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u)+\lambda u^m=f\quad \text{in } \Omega\ (1<p<\infty)$$

provided the condition $m>p-1$ which is the extended version of (3). Other sharp properties on the uniform first term of the expansion of the large solution of (1), for $f\equiv 0$, have been obtained by C. Bandle, G. Díaz, J. García Melián, A. Greco, A. Lazer, S. Kim, N. Kondrat'ev, R. Letelier, J. López-Gómez, M. Marcus, J. Matero, P. McKenna, V. Nikishkin, M. del Pino, G. Porru, J. Sabina and L. Véron among many other authors. We remit to [1] and [2] for some illustrations.

Certainly the geometric properties of the domain can appear in the asymptotic expansion near the boundary. Indeed this influence occurs in secondary terms under more regularity assumptions on the boundary. It is obtained by considering terms containing $\Delta\mathrm{d}(x)$ neglected in the leading coefficient of the expansion. We note the important property

$$\Delta\mathrm{d}(x)=-(\mathrm{N}-1)\mathbf{H}(x),$$

where $\mathbf{H}(x)$ denotes the mean curvature of $\partial\{y\in\Omega:\mathrm{d}(y)<\mathrm{d}(x)\}$ at x (see again [2] or [8]). The simplest geometry is derived on balls, as $\Omega=\mathbf{B}_{\mathrm{R}}(0)$, for which

$$\Delta\mathrm{d}(x)=-\frac{\mathrm{N}-1}{|x|},\quad |x|<\mathrm{R}.$$

The first contribution on this geometrical influence is due to M. del Pino and R. Letelier who proved in [9] that the large solution of (1), for $g(r)=r^m$, $1<m<3$, $\partial\Omega\in\mathcal{C}^4$, $\mathrm{N}>1$ and $f\equiv 0$, admits the expansion

$$u(x)=\left(\frac{2(m+1)}{\lambda(m-1)^2}\right)^{\frac{1}{m-1}}\left(\mathrm{d}(x)\right)^{-\frac{2}{m-1}}\left(1-\left(\frac{(\mathrm{N}-1)\mathbf{H}(x_0)}{m+3}+\mathrm{o}(1)\right)\mathrm{d}(x)\right), \tag{5}$$

where $\mathbf{H}(x_0)$ is the mean curvature of the boundary at the point $x_0\in\partial\Omega$, given by $\mathrm{d}(x)=|x-x_0|$, and $\mathrm{o}(1)\to 0$ as $\mathrm{d}(x)\to 0$. More recently, C. Bandle and M. Marcus have extended the results of [9] by obtaining the dependence on the mean curvature of $\partial\Omega$ in the second order term of the asymptotic behavior of the large solution of (1), again if $f\equiv 0$ (see [2]).

As it was pointed out in the Abstract, the main goal of this paper is to study the whole asymptotic explosive expansion near the boundary of the large solution of (1), here viewed as the source equation

$$-\Delta u+\lambda u^m=f\quad \text{in } \Omega\ (m>1, f\geq 0). \tag{6}$$

As in [3], we will use a simple scheme characterized by means of the behavior

$$f(x)\approx f_0\left(\mathrm{d}(x)\right)^{-q_\tau}\quad \text{as } \mathrm{d}(x)\to 0$$

with

$$\alpha_\tau=\frac{2+\tau}{m-1}\quad\text{and}\quad q_\tau=m\alpha_\tau,\quad (\tau \text{ is a non-negative integer}),$$

for which the *low explosive sources* are given by $\tau=0$ and $f_0\geq 0$ and the *high explosive sources* by $\tau>0$ and $f_0>0$. We note that large solutions for low explosive sources have been considered in the literature, mainly for null sources $f\equiv 0$ (see the above references). On the other hand, to the best of our knowledge only in [3, Theorem 3.8] large solutions for high explosive sources have been studied.

So that, our main contribution is sketched as follows (see Theorem 1). Let us assume $\partial\Omega$ smooth enough and $f\in\mathcal{C}(\Omega)$, $f\geq 0$, verifying

$$f(x)=\left(\mathrm{d}(x)\right)^{-q_\tau}\left(f_0+\sum_{n=1}^{\mathrm{M}_\tau}f_n\left(\mathrm{d}(x)\right)^n\right),\quad x\in\Omega_{\delta_0},$$