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Bounded traveling waves of the Burgers–Huxley equation[☆]

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ABSTRACT

In order to investigate bounded traveling waves of the Burgers–Huxley equation, bifurcations of codimension 1 and 2 are discussed for its traveling wave system. By reduction to center manifolds and normal forms we give conditions for the appearance of homoclinic solutions, heteroclinic solutions and periodic solutions, which correspondingly give conditions of existence for solitary waves, kink waves and periodic waves, three basic types of bounded traveling waves. Furthermore, their evolutions are discussed to investigate the existence of other types of bounded traveling waves, such as the oscillatory traveling waves corresponding to connections between an equilibrium and a periodic orbit and the oscillatory kink waves corresponding to connections of saddle–focus.

1. Introduction

Being a reduced object of the Navier–Stokes equation to the case of space dimension 1, the Burgers equation [1]

$$u_t + auu_x - u_{xx} = 0,$$

where constant a controls the nonlinearity, is regarded as an elementary partial differential equation modeling the far field wave propagation in a nonlinear dissipative system. It can be used to describe many physical phenomena, such as sound waves in viscous media, magnetohydrodynamic waves in media with finite electrical conductivity, etc. In 1986, Satsuma [2] modified it further, by adding a nonlinear reaction term, to the form

$$u_t + auu_x - u_{xx} + bu(u-1)(u-r) = 0, \tag{1.1}$$

called the Burgers–Huxley equation (see [3]), where the constant b describes a nonlinear source. Eq. (1.1) includes several known evolution equations, e.g., the FitzHugh–Nagumo equation [4] when $a = 0$ and the Newell–Whitehead equation [5] when $a = 0$ and $r = -1$, and models more extensive classes of wave propagation in biological and chemical systems.

To understand complicated nonlinear wave phenomena, traveling wave solutions of a PDE play important roles. In general, three basic types of bounded traveling waves can occur for a PDE; these are periodic waves, kink waves and solitary waves. They are also known, respectively, as periodic wave trains, fronts, and pulses. Among them, the soliton, as a special type of solitary wave, is of great importance because it possesses the properties of both a wave and a particle. It is used widely

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in many fields and has been studied deeply by many people (see [6–10]). The Burgers–Huxley equation (1.1) exhibits many nonlinear behaviors, that are also reflected partially in its traveling waves. Therefore, its traveling waves are frequently studied. In 1986, assuming $0 < r < 1$, Satsuma [2] reduced the equation to a bilinear form and obtained kink wave solutions. Later, Wang et al. [11] applied nonlinear transformations to obtain generalized kink wave solutions of Eq. (1.1) and its generalized form. More exact solutions, including various kink wave solutions and periodic solutions with respect to spatial variable x, were obtained in [12] by a reduction with the Cole–Hopf transformation. In addition, symmetries were investigated by Estévez and Gordoa [13,14] in the 1990's. Since Eq. (1.1) is not an exactly solvable equation, because it does not pass the Painlevé test [15], in recent years more attention has been paid to numerical methods, such as the homotopy analysis method (HAM) [16] and the spectral collocation method [17], to solve it and its generalized form.

Although there have been some profound results concerning the traveling waves of Eq. (1.1) which have contributed to our understanding of nonlinear physical phenomena and wave propagation, there are still some unresolved problems because Eq. (1.1) is not an exactly solvable equation. For instance, can any other type of bounded traveling wave solutions occur for Eq. (1.1)? If they exist, what dynamical behavior do they have? If these problems can be solved, the study of both exact traveling wave solutions and numerical solutions of Eq. (1.1) will be promoted further. In fact, these problems have involved bifurcation problems. Recall that heteroclinic orbits are trajectories which have two distinct equilibria as their α and ω-limit sets, and homoclinic orbits are trajectories whose α and ω-limit sets consist of the same equilibrium. So, the three basic types of bounded traveling waves mentioned above correspond to periodic, heteroclinic, and homoclinic orbits of the traveling wave system of a PDE respectively (see [18,19]). It is just this relationship that makes the bifurcation theory of dynamical system become an effective method to investigate bifurcations of traveling waves of PDEs. In recent decades, much effort has been devoted to bifurcations of traveling waves of PDEs. In 1997 Peterhof et al. [20] investigated the persistence and continuation of exponential dichotomies for solitary wave solutions of semilinear elliptic equations on infinite cylinders so that a Lyapunov–Schmidt reduction can be applied near solitary waves. Sánchez-Garduño and Maini [21] considered the existence of one-dimensional traveling wave solutions in nonlinear diffusion degenerate Nagumo equations and employed a dynamical systems approach to prove the bifurcation of a heteroclinic cycle. Later Katzengruber et al. [18] analyzed bifurcations of traveling waves, such as the Hopf bifurcation, multiple periodic orbit bifurcation, homoclinic bifurcation and heteroclinic bifurcation, in a standard model of electrical conduction in extrinsic semiconductors, which in scaled variables is actually a singular perturbation problem of a 3-dimensional ODE system. In 2002 Constantin and Strauss [22] constructed periodic traveling waves with vorticity for the classical inviscid water wave problem under the influence of gravity, described by the Euler equation with a free surface over a flat bottom, and used global bifurcation theory to construct a connected set of such solutions, containing flat waves as well as waves that approach flows with stagnation points. In 2003 Huang et al. [23] employed the Hopf bifurcation theorem to establish the existence of traveling front solutions and small amplitude traveling wave train solutions for a reaction–diffusion system based on a predator–prey model, which are equivalent to heteroclinic orbits and small amplitude periodic orbits in $\mathbb{R}^4$ respectively. Besides these, many results on the bifurcations of traveling waves for the Camassa–Holm equation, KdV equation and Zhiber–Shabat equation can be found in [24–27].

Motivated by the above results and enlightened by [18], for the more general case $ab \neq 0, r \in \mathbb{R}$, we try to give existence conditions and investigate the evolution behaviour of bounded traveling waves of Eq. (1.1) by studying bifurcations of its traveling wave solutions. Overcoming the difficulties of nonintegrability, we exhibit bifurcations of codimension 1 and 2 of the traveling wave system of Eq. (1.1), including the Hopf bifurcation, transcritical bifurcation, Bogdanov–Takens bifurcation, homoclinic orbits bifurcation, heteroclinic orbits bifurcation and Poincaré bifurcation, by computing center manifolds, normal forms and Abelian integrals. Using these bifurcation results and the relationships [18,19] between bounded traveling wave solutions and bounded orbits of the corresponding traveling wave system of Eq. (1.1), we obtain the existence conditions of three basic types of traveling wave solutions, including kink wave, solitary wave and periodic wave solutions. Furthermore, by discussing their evolution, some other bounded traveling wave solutions are found. In summary, for the Burgers–Huxley equation, there exist many types of bounded traveling wave solutions, including a type of monotone kink wave solution corresponding to connections of saddle–saddle, a type of oscillatory kink wave solution corresponding to connections of saddle–focus, solitary wave solutions corresponding homoclinic orbits, periodic wave solutions corresponding to periodic orbits and a type of oscillatory traveling wave solution corresponding to connections between an equilibrium and a periodic orbit.

2. Hopf bifurcation

It is well known that a traveling wave solution is a special solution of the form $u(x, t) = u(x - ct)$, where $c \neq 0$ is the wave velocity. So, we can make the transformation $\xi = x - ct$ to change Eq. (1.1) into the traveling wave system

$$-u'' - cu' + auu' + bu(u - 1)(u - r) = 0,$$

which has the equivalent form

$$\begin{cases} u' = v, \\ v' = -cv + auv + bu(u - 1)(u - r), \end{cases} \tag{2.2}$$

where $'$ denotes $d/d\xi$. All equilibria of system (2.2) are of the form $(u_0, 0)$, where u_0 satisfies

$$u_0(u_0 - 1)(u_0 - r) = 0.$$