Nonlinear Analysis 74 (2011) 1402–1409

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Blow-up of solutions for a nonlinear beam equation with fractional feedback

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ARTICLE INFO

Article history:
Received 8 May 2010
Accepted 4 October 2010

MSC:
35L10
93B30

Keywords:
Blow up
Boundary feedback
Fractional derivative
Non-dissipative system
Singular kernel

ABSTRACT

A nonlinear beam equation describing the transversal vibrations of a beam with boundary feedback is considered. The boundary feedback involves a fractional derivative. We discuss the asymptotic behavior of solutions. In fact, we prove that solutions blow up in finite time under certain assumptions on the nonlinearity.

1. Introduction and description of the model

In this paper we study the behavior of solutions to the following nonlinear beam equation with fractional damping at the boundary

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta g(\Delta u) = 0, \quad x \in \Omega, t > 0 \\ u = \dfrac{\partial u}{\partial \nu} = 0, \quad x \in \Gamma_1,\ t > 0 \\ \Delta u = 0, \quad x \in \Gamma_0,\ t > 0 \\ \dfrac{\partial \Delta u}{\partial \nu} = \dfrac{c}{\Gamma(\beta)} \displaystyle\int_0^t (t-s)^{\beta-1} u_t \mathrm{d}s - au, \quad x \in \Gamma_0,\ t > 0 \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. The boundary is divided into Γ_0 and Γ_1 in such a way that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\lambda_{n-1}(\Gamma_1) > 0$ where λ_{n-1} denotes the $(n-1)$-dimensional Lebesgue measure on the boundary $\partial\Omega$. The initial data $u_0(x)$ and $u_1(x)$ are given functions, $g(s)$ is a given nonlinear function, $\partial/\partial\nu$ denotes the outward normal derivative and $\Gamma(.)$ is the usual Euler gamma function. The constants a and c are positive and the power β in the integral term is such that $0 < \beta < 1$.

This problem is known as the Euler–Bernoulli beam problem and describes the transversal vibrations of a beam. Here the control is a torque applied on a part of the boundary of the beam. The integral term in the boundary condition is a time fractional derivative of u of order $1-\beta$. It represents a boundary feedback which helps reduce the effect of the reflected waves. In fact, it is a boundary damping. Therefore, it is important to have an idea of the sufficient conditions which make

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doi:10.1016/j.na.2010.10.012

the nonlinear source take it over this dissipation and drive the system to blow up in finite time. We recall that the fractional derivative of $w(t)$ of order α in the sense of Caputo (see [1]) is defined by

$$\partial_t^{\alpha} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}s} w(s)\mathrm{d}s, \quad 0 < \alpha < 1$$

provided that the integral exists. The reader is referred to the books [2,1,3] for more details about derivatives and integrals of fractional order.

The well-posedness as well as the stabilization of the Euler–Bernoulli beam with this control (but without the nonlinearity, i.e. $g = 0$), have been discussed in [4]. The present authors proved an exponential growth result for problem (1) with a polynomial source (instead of the term $\Delta g(\Delta u)$) at the boundary in [5]. Furthermore, they studied the case when the polynomial source acts on all of the domain and proved a blow-up result in [6]. There are several results more or less related to the present problem. We refer the reader to [7–18] and the references cited therein. It is worth mentioning the works [19,20] where blow up results are established for problems with internal dampings and similar nonlinearities as ours.

In the present paper, we prove blow-up of solutions in finite time to problem (1). To this end, we appeal to the method used in [20] instead of the familiar "concavity method" [21,22]. For that, we need to establish a differential inequality $y'' + y' \geq ct^{1-m}(y' + y)^{\alpha}$, where c is a positive constant, $1 < m \leq 2, \alpha > 1$, for a suitable twice differentiable function $y(t)$.

Our paper is organized as follows: In Section 2, we present some notations and lemmas which will be needed in the sequel. In Section 3, we state and prove the blow-up result.

2. Preliminaries

In this section we recall some definitions and lemmas which will be useful in deriving our result. We begin this section with an existence and uniqueness theorem for the problem (1).

Theorem 1. *Suppose that $u_0 \in H^2_{\Gamma_1}(\Omega), u_1 \in L^2(\Omega)$ and $g \in C^2(\mathbf{R})$. Then the problem (1) has a unique weak solution $u \in C([0, T); H^2_{\Gamma_1}(\Omega)) \cap C^1([0, T); L^2(\Omega))$, where $[0, T)$ is a maximal time interval and $H^2_{\Gamma_1}(\Omega)$ denotes*

$$H^2_{\Gamma_1}(\Omega) := \left\{ w \in H^2(\Omega) : w = \partial w/\partial \nu = 0 \text{ on } \Gamma_1 \right\}.$$

This theorem can be proved using a similar argument to the one in [13–15,4,9] (for the case $n = 1$ see [15,4]). Let us define the classical energy associated to problem (1) by

$$E(t) := \frac{1}{2}\int_{\Omega} (u_t^2 + |\Delta u|^2)\mathrm{d}x + \int_{\Omega} G(\Delta u)\mathrm{d}x - \frac{a}{2}\int_{\Gamma_0} |u|^2 \mathrm{d}\sigma,$$

where $G(t) = \int_0^t g(s)\mathrm{d}s$. We will assume, without loss of generality, that $a = 1$ and $c = 1$. Multiplying the equation in (1) by u_t and integrating over Ω we obtain

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\frac{1}{\Gamma(\beta)}\int_{\Gamma_0} u_t \int_0^t (t-s)^{\beta-1} u_t \mathrm{d}s\mathrm{d}\sigma.$$

Replacing t by s and s by z, then integrating from 0 to t, we get

$$E(t) - E(0) = -\frac{1}{\Gamma(\beta)} \int_0^t \int_{\Gamma_0} u_t \int_0^s (s-z)^{\beta-1} u_z(z)\mathrm{d}z\mathrm{d}\sigma\mathrm{d}s. \tag{2}$$

Clearly,

$$E(t) \leq E(0) \quad \text{for all } t \geq 0, \tag{3}$$

because $t^{\beta-1}, 0 < \beta < 1$ is a positive definite function.

Lemma 1 (*Young Inequality, See [23]*)**.** *Let $f \in L^p(\mathbf{R})$ and $g \in L^q(\mathbf{R})$ with $1 \leq p, q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$. Then $f * g \in L^r(\mathbf{R})$ and*

$$\|f * g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}.$$

Lemma 2 (*See [24]*)**.** *Let Ω be a regular and bounded domain and define the Hilbert space $H^1_{\Gamma_1}(\Omega)$ by*

$$H^1_{\Gamma_1}(\Omega) = \left\{ u \in H^1(\Omega), u|_{\Gamma_1} = 0 \right\}.$$

Then,

$$H^1_{\Gamma_1}(\Omega) \hookrightarrow L^p(\Gamma_0) \quad \text{for } 2 \leq p < r$$

with $r = \begin{cases} \frac{2(n-1)}{n-2} & \text{if } n \geq 3 \\ +\infty & \text{if } n = 1, 2. \end{cases}$