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Blowup for the Euler and Euler–Poisson equations with repulsive forces

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ABSTRACT

In this paper, we study the blowup of the N-dim Euler or Euler–Poisson equations with repulsive forces, in radial symmetry. We provide a novel integration method to show that the non-trivial classical solutions (ρ, V), with compact support in $[0, R]$, where $R > 0$ is a positive constant and in the sense which $\rho(t, r) = 0$ and $V(t, r) = 0$ for $r \geq R$, under the initial condition

$$H_0 = \int_0^R rV_0 \mathrm{d}r > 0, \tag{1}$$

blow up on or before the finite time $T = R^3/H_0$ for pressureless fluids or $\gamma > 1$.

The main contribution of this article provides the blowup results of the Euler ($\delta = 0$) or Euler–Poisson ($\delta = 1$) equations with repulsive forces, and with pressure ($\gamma > 1$), as the previous blowup papers (Makino et al., 1987 [18], Makino and Perthame, 1990 [19], Perthame, 1990 [20] and Chae and Tadmor, 2008 [24]) cannot handle the systems with the pressure term, for C^1 solutions.

1. Introduction

The isentropic Euler ($\delta = 0$) or Euler–Poisson ($\delta = \pm 1$) equations can be written in the following form:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho[u_t + (u \cdot \nabla)u] + \nabla P = \rho \nabla \Phi, \\ \Delta \Phi(t, x) = \delta \alpha(N) \rho, \end{cases} \tag{2}$$

where $\alpha(N)$ is a constant related to the unit ball in R^N: $\alpha(1) = 1$, $\alpha(2) = 2\pi$ and $\alpha(3) = 4\pi$. And as usual, $\rho = \rho(t, x) \geq 0$ and $u = u(t, x) \in \mathbf{R}^N$ are the density and the velocity respectively. $P = P(\rho)$ is the pressure function. The γ-law can be applied on the pressure term $P(\rho)$, i.e.

$$P(\rho) = K\rho^{\gamma}, \tag{3}$$

which is a common hypothesis. If the parameter is set as $K > 0$, we call the system *with pressure*; if $K = 0$, we call it *pressureless*. The constant $\gamma = c_P/c_v \geq 1$, where c_P, c_v are the specific heats per unit mass under constant pressure and constant volume respectively, is the ratio of the specific heats, that is, the adiabatic exponent in Eq. (3). In particular, the fluid is called isothermal if $\gamma = 1$. If $K > 0$, we call the system with pressure; if $K = 0$, we call it pressureless.

In the above systems, the self-gravitational potential field $\Phi = \Phi(t, x)$ is determined by the density ρ itself, through the Poisson equation $(2)_3$.

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When $\delta = -1$, the system can model fluids that are self-gravitating, such as gaseous stars. In addition, the evolution of the simple cosmology can be modelled by the dust distribution without the pressure term. This describes the stellar systems of collisionless and gravitational n-body systems [1]. And the pressureless Euler–Poisson equations can be derived from the Vlasov–Poisson–Boltzmann model with the zero mean free path [2]. For $N = 3$ and $\delta = -1$, Eq. (2) are the classical (non-relativistic) descriptions of a galaxy in astrophysics. See [3,4], for details about the systems.

When $\delta = 1$, the system is the compressible Euler–Poisson equations with repulsive forces. Eq. $(2)_3$ is the Poisson equation through which the potential with repulsive forces is determined by the density distribution of the electrons. In this case, the system can be viewed as a semiconductor model. See [5,6] for detailed analysis of the system.

On the other hand, the Poisson equation $(2)_3$ can be solved as

$$\Phi(t,x) = \delta \int_{R^N} G(x-y)\rho(t,y)\mathrm{d}y, \tag{4}$$

where G is Green's function for the Poisson equation in the N-dimensional spaces defined by

$$G(x) \doteq \begin{cases} |x|, & N = 1; \\ \log|x|, & N = 2; \\ \dfrac{-1}{|x|^{N-2}}, & N \geq 3. \end{cases} \tag{5}$$

Usually, the Euler–Poisson equations can be rewritten in the scalar form:

$$\begin{cases} \dfrac{\partial\rho}{\partial t} + \displaystyle\sum_{k=1}^{N} u_k \dfrac{\partial\rho}{\partial x_k} + \rho \sum_{k=1}^{N} \dfrac{\partial u_k}{\partial x_k} = 0, \\ \rho\left(\dfrac{\partial u_i}{\partial t} + \displaystyle\sum_{k=1}^{N} u_k \dfrac{\partial u_i}{\partial x_k}\right) + \dfrac{\partial P}{\partial x_i} = \rho \dfrac{\partial\Phi}{\partial x_i}, \quad \text{for } i = 1, 2, \ldots, N. \end{cases} \tag{6}$$

For the construction of the analytical solutions for the systems, interested readers should refer to [7–11]. The results for local existence theories can be found in [12–14]. The analysis of stabilities for the systems may be referred to [15–21,9, 22–25].

We seek the radial symmetry solutions

$$\rho(t,\vec{x}) = \rho(t,r) \quad \text{and} \quad \vec{u} = \frac{\vec{x}}{r}V(t,r) =: \frac{\vec{x}}{r}V, \tag{7}$$

with the radius $r = \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}$.

For the solutions in spherical symmetry, the Poisson equation $(2)_3$ is transformed to

$$r^{N-1}\Phi_{rr}(t,x) + (N-1)\,r^{N-2}\Phi_r = \alpha(N)\delta\rho r^{N-1}, \tag{8}$$

$$\Phi_r = \frac{\alpha(N)\delta}{r^{N-1}} \int_0^r \rho(t,s)s^{N-1}\mathrm{d}s. \tag{9}$$

By standard computation, the Euler–Poisson equations in radial symmetry can be written in the following form:

$$\begin{cases} \rho_t + V\rho_r + \rho V_r + \dfrac{N-1}{r}\rho V = 0, \\ \rho\left(V_t + VV_r\right) + P_r(\rho) = \rho\Phi_r(\rho). \end{cases} \tag{10}$$

Historically, Makino et al. initially defined the tame solutions [18] for outside the compact of the solutions

$$V_t + VV_r = 0. \tag{11}$$

Following this, Makino and Perthame considered the tame solutions for the system with gravitational forces [19]. After that Perthame discovered the blowup results for 3-dimensional pressureless system with repulsive forces [20] ($\delta = 1$). In short, all the results above rely on the solutions with radial symmetry:

$$V_t + VV_r = \frac{\alpha(N)\delta}{r^{N-1}} \int_0^r \rho(t,s)s^{N-1}\mathrm{d}s. \tag{12}$$

And the Emden ordinary differential equations were deduced on the boundary point of the solutions with compact support:

$$\frac{\mathrm{D}^2 R}{\mathrm{D}t^2} = \frac{\delta M}{R^{N-1}}, \qquad R(0,R_0) = R_0 \geq 0, \qquad \dot{R}(0,R_0) = 0, \tag{13}$$