Nonlinear Analysis 72 (2010) 635–650

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Generalized Glimm scheme to the initial boundary value problem of hyperbolic systems of balance laws

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ARTICLE INFO

Article history:
Received 4 May 2009
Accepted 2 July 2009

MSC:
35L60
35L65
35L67

Keywords:
Hyperbolic systems of balance laws
Riemann problem
Boundary Riemann problem
Lax's method
Generalized Glimm scheme

ABSTRACT

In this paper we provide a generalized version of the Glimm scheme to establish the global existence of weak solutions to the initial-boundary value problem of 2×2 hyperbolic systems of conservation laws with source terms. We extend the methods in [J.B. Goodman, Initial boundary value problem for hyperbolic systems of conservation laws, Ph.D. Dissertation, Stanford University, 1982; J.M. Hong, An extension of Glimm's method to inhomogeneous strictly hyperbolic systems of conservation laws by "weaker than weak" solutions of the Riemann problem, J. Differential Equations 222 (2006) 515–549] to construct the approximate solutions of Riemann and boundary Riemann problems, which can be adopted as the building block of approximate solutions for our initial-boundary value problem. By extending the results in [J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, Comm. Pure Appl. Math. 18 (1965) 697–715] and showing the weak convergence of residuals, we obtain stability and consistency of the scheme.

1. Introduction

We consider the following 2×2 nonlinear hyperbolic system of balance laws

$$u_t + f(a, u)_x = a' g(a, u), \tag{1.1}$$

where $u = u(x, t) = (v(x, t), w(x, t)), f(a, u) = (f_1(a, u), f_2(a, u)), g(a, u) = (g_1(a, u), g_2(a, u))$ are smooth functions of (a, u), $a = a(x)$ is a Lipschitz continuous function of x which is of finite total variation and $a' = \frac{da}{dx}$. Following the lead by Isaacson–Temple [1] and LeFloch–Liu [2], we augment system (1.1) by adding the equation $a_t = 0$ and obtain the following 3×3 hyperbolic system of balance laws

$$U_t + F(U)_x = a' G(U), \tag{1.2}$$

where $U = (a, v, w)$, $F(U) = (0, f(a, u))$ and $G(U) = (0, g(a, u))$.

In this paper we investigate the global existence of weak solutions to the initial-boundary value problem of (1.2):

$$\begin{cases} U_t + F(U)_x = a' G(U), & (x, t) \in (0, \infty) \times (0, \infty), \\ U(x, 0) = U_0(x), & x \geq 0, \\ v(0, t) = v_B(t), & t \geq 0, \end{cases} \tag{1.3}$$

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doi:10.1016/j.na.2009.07.003

where $U_0(x) \equiv (a(x), v_0(x), w_0(x))$, $v_B(t)$ are bounded functions with finite total variations and satisfy $v_B(0) = v_0(0)$. We assume that system (1.2) is strictly hyperbolic, that is, the eigenvalues $\lambda_0 = 0$, λ_1 and λ_2 of Jacobian matrix $DF(U)$ are real and distinct. Moreover, each characteristic field of (1.2) is either genuinely nonlinear or linear degenerate [3]. We assume further that there is an open region $\Omega \subset \mathbb{R}^3$ such that f, g satisfy

(A_1) each component of $(\frac{\partial f}{\partial u})^{-1}(g - \frac{\partial f}{\partial a})(U)$ is nonzero for $U \in \Omega$,
(A_2) $\frac{\partial f_1}{\partial w}(U) \neq 0$ for $U \in \Omega$.

An important application of (1.1) is the one-dimensional compressible Euler equations in a variable area duct

$$\begin{cases} \rho_t + (\rho u)_x = -(a'/a)\rho u, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = -(a'/a)\rho u^2, \end{cases} \tag{1.4}$$

where ρ, u and $p(\rho)$ are the density, velocity and pressure of the fluid respectively, and $a(x)$ represents the area of cross-section of the duct.

We review some previous results to this topic. The entropy solutions to the Riemann problem of

$$U_t + F(U)_x = 0 \tag{1.5}$$

was first studied by Lax [4,28]. The solutions are self-similar functions consisting of constant states separated by elementary waves (rarefaction waves, shock waves and contact discontinuities). To the Cauchy problem of (1.5), the global existence of weak solutions was established by Glimm [5] when initial data is uniformly bounded and of small total variation. For the initial-boundary value problem (IBVP for short), Nishida and Smoller [6] studied the piston and double piston problems and obtained the global existence of weak solutions. On the other hand, the IBVP of unsteady flow in gas dynamics was studied by Liu [7]. Moreover, the IBVP of (1.5) with arbitrary shape of boundary was first studied by Goodman [8]. He proved the global existence of weak solutions when the initial and boundary data satisfy the so-called smallness and non-degeneracy conditions.

To the quasilinear hyperbolic system

$$U_t + F(x, U)_x = G(x, U), \tag{1.6}$$

the global existence results for Cauchy problem was established by Liu [9]. We also remark that for the well-posedness theory for initial-boundary value problems, we refer the reader to [10–14]. On the other hand, the hyperbolic system of balance laws

$$U_t + F(a(x), U)_x = a'G(a(x), U), \tag{1.7}$$

was studied by Hong [15]. In [15], the global existence of weak solutions to Cauchy problem of (1.7) was obtained by the generalized Glimm scheme based on "weaker than weak" solutions of the Riemann problem. To the non-strictly hyperbolic (resonant) systems, we refer the readers to the results in [16–18,26,27]. Note that (1.7) can also be written as a system in non-conservative form. The global existence and stability of measure-valued solutions to non-conservative systems were studied by LeFloch [19,20], LeFloch–Liu [2] and Dal Maso–LeFloch–Murat [21]. To general quasilinear hyperbolic system

$$U_t + F(U, x, t)_x = G(U, x, t), \tag{1.8}$$

the results for global existence of weak solutions to the Cauchy problem of (1.8) can be found in [22,23,29].

We notice that the results described above for the global existence of weak solutions to IBVPs are restricted to the hyperbolic systems without source terms. Recently, in the paper by Hong–Hsu–Su [24], the authors studied the following IBVP of quasilinear wave equation

$$\begin{cases} u_{tt} - \big(p(\rho(x), u_x)\big)_x = \rho(x)h(\rho(x), u, u_x), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = w_0(x), \\ u_x(0, t) = v_B(t), \end{cases} \tag{1.9}$$

or equivalently

$$\begin{cases} v_t - w_x = 0, \\ w_t - \big(p(\rho(x), v)\big)_x = \rho(x)h(\rho(x), u, v), \\ w(x, 0) = w_0(x), \quad v(0, t) = v_B(t), \quad u(x, 0) = u_0(x), \end{cases} \tag{1.10}$$

where $v = u_x$ and $w = u_t$. By Glimm scheme and the perturbation technique for approximate solutions near the boundary, they obtained the global existence of weak solutions to (1.10) under the condition that the first and second derivatives of source $h(\rho, u, v)$ to be uniformly bounded. In this paper, the generalized Glimm scheme we use enables us to relax the conditions of f and g given in [24] to establish the global existence results. More precisely, we only require condition (A_2) on f to construct the approximate solutions near the boundary. Furthermore, since the solutions constructed for the boundary Riemann problem are exact solutions, we do not need to impose any condition on the second derivatives of source term.

Now we give the definition for the weak solutions of IBVP (1.3), and state the main theorem of this paper.