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Existence and nonexistence of ground state solutions for elliptic equations with a convection term[☆]

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ABSTRACT

We deal with the existence of positive solutions u decaying to zero at infinity, for a class of equations of Lane–Emden–Fowler type involving a gradient term. One of the main points is that the differential equation contains a semilinear term $\sigma(u)$ where $\sigma : (0,\infty) \to (0,\infty)$ is a smooth function which can be both unbounded at infinity and singular at zero. Our technique explores symmetry arguments as well as lower and upper solutions.

1. Introduction

We shall establish results on existence and nonexistence of solutions for the nonlinear elliptic problem

$$\begin{cases} -\Delta u = \lambda a(x)\left(\sigma(u) + |\nabla u|^q\right) & \text{in } \mathbf{R}^N, \\ u > 0 \quad \text{in } \mathbf{R}^N, & u(x) \overset{|x|\to\infty}{\longrightarrow} 0, \end{cases} \tag{1.1}$$

where $N \geq 3$, $0 < q < 1$ and $\lambda > 0$ is a parameter, while the so-called anisotropic potential function $a : \mathbf{R}^N \to (0,\infty)$ is locally Hölder continuous and the nonlinearity $\sigma : (0,\infty) \to (0,\infty)$ is C^1 and both possibly singular at the origin and strongly unbounded at infinity.

The following conditions shall be assumed on σ:

$$\text{(i)} \quad \frac{\sigma(s)}{s} \overset{s\to 0}{\to} \sigma_0, \qquad \text{(ii)} \quad \frac{\sigma(s)}{s} \overset{s\to\infty}{\to} \sigma_\infty, \tag{1.2}$$

where $0 < \sigma_0 \leq \infty$ and $0 \leq \sigma_\infty \leq \infty$.

When compared to recent results on the subject, some of which are recalled below, no monotonicity will be imposed on either σ or the quotient function $\sigma(s)/s$ in the present paper. Moreover, the case $\sigma_0 = \sigma_\infty = \infty$ is also treated here.

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Let ρ be the function defined by

$$\rho(r) = \max_{|x|=r} a(x), \quad r \geq 0.$$

The well known conditions

$$\int_0^\infty r\rho(r)\mathrm{d}r < \infty \tag{1.3}$$

and

$$\int_0^\infty r\rho(r)\mathrm{d}r = \infty, \tag{1.4}$$

are crucial respectively in the proof of existence and nonexistence of solutions. In this regard, condition (1.3) is in fact nearly necessary for existence, see [1,2] and references therein.

The main equation in (1.1) appears in the physical sciences and is referred to in the literature as the Lane–Emden–Fowler equation, cf. [3–5] and references therein. We recall below a few recent results more closely related to our interest in the present paper.

It was proved in [6] that if (1.3) holds then the problem

$$\begin{cases} -\Delta u = a(x)\left(g(u) + f(u) + |\nabla u|^q\right) & \text{in } \mathbf{R}^N, \\ u > 0 \quad \text{in } \mathbf{R}^N, & u(x) \overset{|x|\to\infty}{\to} 0, \end{cases} \tag{1.5}$$

admits an entire solution if $g : (0, \infty) \to (0, \infty)$ is a decreasing C^1-function such that $g(s) \overset{s\to 0}{\to} \infty$ and $f : [0, \infty) \to [0, \infty)$ is a Hölder continuous function satisfying

$$sf(s) > 0 \quad \text{for } s > 0, \qquad f(s)/s \quad \text{is non-increasing},$$

$$f(s)/s \overset{s\to 0}{\longrightarrow} \infty, \qquad f(s)/s \overset{s\to\infty}{\longrightarrow} 0.$$

Typical examples of functions g, f satisfying the conditions in [6] are

$$g : (0, \infty) \to (0, \infty), \qquad g(s) = s^{-\beta}, \quad \text{where } 0 < \beta < \infty,$$
$$f : [0, \infty) \to [0, \infty), \qquad f(s) = s^\gamma, \quad \text{where } 0 < \gamma < 1.$$

It was recently showed in [7], that if (1.3) holds then (1.5) admits an entire solution under the following set of conditions,

$$g : (0, \infty) \to (0, \infty) \quad \text{is } C^1, \qquad g(s)/s \overset{s\to 0}{\longrightarrow} \infty, \qquad g(s)/s \overset{s\to\infty}{\longrightarrow} 0,$$

$$f : [0, \infty) \to [0, \infty) \quad \text{is Hölder continuous}, \qquad f(s)/s \overset{s\to 0}{\longrightarrow} \infty, \qquad f(s)/s \overset{s\to\infty}{\longrightarrow} 0.$$

Examples of functions satisfying the conditions in [7] are,

$$g : (0, \infty) \to (0, \infty), \qquad g(s) = s^{-\beta} + \sin(s) + 1, \quad \text{where } 0 < \beta < \infty,$$
$$f : [0, \infty) \to [0, \infty), \qquad f(s) = s^\gamma + \sin(s) + 1, \quad \text{where } 0 < \gamma < 1.$$

Next we provide some examples to which the results of the present paper apply. To this end consider the class of functions

$$\sigma : (0, \infty) \to (0, \infty), \qquad \sigma(s) = s^{-\beta} + (1 + \sin(s)) + s^\gamma,$$

where β and γ are positive constants.

We point out that

(i) if $\gamma \in (0, 1)$ then $\sigma_\infty = 0$,

(ii) if $\gamma = 1$ then $\sigma_\infty = 1$,

(iii) if $\gamma > 1$ then $\sigma_\infty = \infty$.

Setting

$$\sigma(s) = g(s) + f(s), \quad s > 0,$$

where f is extended continuously to zero, none of the cases above is covered by [6], while the items (ii)–(iii) are not covered by the results in [7]. However as we will see below all of those results are covered by our Theorem 1.1.

There is by now a broad literature and we refer the reader to [8–13,4,14–20] and their references.

In order to state our main results let

$$\zeta_1(a, B) := \inf_{\{w\in W_0^{1,2}(B), w\neq 0\}} \left\{ \frac{\int_B |\nabla w|^2 \mathrm{d}x}{\int_B a(x)|w|^2 \mathrm{d}x} \right\}, \tag{1.6}$$