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Optimal control of the viscous weakly dispersive Degasperis–Procesi equation

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ABSTRACT

In this paper, we study the optimal control problem for the viscous weakly dispersive Degasperis–Procesi equation. We deduce the existence and uniqueness of a weak solution to this equation in a short interval by using the Galerkin method. Then, according to optimal control theories and distributed parameter system control theories, the optimal control of the viscous weakly dispersive Degasperis–Procesi equation under boundary conditions is given and the existence of an optimal solution to the viscous weakly dispersive Degasperis–Procesi equation is proved.

1. Introduction

Recently, Holm and Staley [1] introduced the b-family PDEs that described the balance between convection and stretching for small viscosity in the dynamics of one dimensional nonlinear wave in fluids

$$m_t + \underbrace{um_x}_{convection} + \underbrace{bu_x m}_{stretching} = \underbrace{\varepsilon m_{xx}}_{viscosity}, \tag{1.1}$$

where $u = g * m$ denotes $u(x) = \int_{-\infty}^{\infty} g(x-y)m(y)\mathrm{d}y$. The convolution relates velocity u to momentum density m by integration against the kernel $g(x)$.

When Eq. (1.1) is restricted to the peakon case $g(x) = \mathrm{e}^{-|x|/\alpha}$ with length scale α and $m = u - \alpha^2 u_{xx}$, it may be expressed solely in terms of the velocity $u(x,t)$ as (see [1])

$$u_t - \alpha^2 u_{xxt} - \varepsilon(u_{xx} - u_{xxxx}) + (b+1)uu_x = \alpha^2(bu_x u_{xx} + uu_{xxx}), \tag{1.2}$$

where b, α and ε are arbitrary real constants. Holm and Staley studied the effects of the balance parameter b and kernel $g(x)$ of solitary wave structures and investigated their interactions analytically for $\varepsilon = 0$ and numerically for small viscosity $\varepsilon \neq 0$, of [1].

With $\varepsilon = 0$ in Eq. (1.2), it becomes the usual b-equation

$$u_t - \alpha^2 u_{xxt} + (b+1)uu_x = \alpha^2(bu_x u_{xx} + uu_{xxx}). \tag{1.3}$$

The b-equation (1.3) can be derived as the family of asymptotically equivalent shallow water wave equations that emerge at quadratic order accuracy for any $b \neq -1$ by an appropriate Kodama transformation, of [2,3]. For the case $b = -1$, the

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corresponding Kodama transformation is singular and the asymptotic ordering is violated, of [2,3]. The solutions of the b-equation (1.3) were studied numerically for various values of b in [1,4], where b was taken as a bifurcation parameter. The KdV equation, the Camassa–Holm equation, and the Degasperis–Procesi equation are the only three integrable equations in the b-equation (1.3), which was shown in [5,6] by using Painleve analysis. The b-equation (1.3) admits peakon solutions for any $b \in \mathbb{R}$, of [1,4,6]. The Cauchy problem for Eq. (1.3) with $\alpha \neq 0$ on the line has been discussed recently in [7]. The local well-posedness for the b-equation, a precise blowup scenario, several blowup results and global existence result of strong solutions, and the uniqueness and the existence of global weak solution to the b-equation on the line have been proved in [7]. Recently, it was pointed out that the KdV equation and the Camassa–Holm equation could be relevant to the modeling of tsunami waves in [8–11].

If $\alpha = 0$, $b = 2$ and $\varepsilon = 0$, then Eq. (1.2) becomes the well-known Korteweg-de Vries equation which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity, of [12]. In this model $u(t, x)$ represents the wave height above a flat bottom, where x is proportional to the distance in the direction of propagation and t is proportional to the elapsed time. The KdV equation is completely integrable and its solitary waves are solitons [13,14]. The Cauchy problem of the KdV equation has been the subject of a number of studies, and a satisfactory local or global (in time) existence theory is now in hand (for example, see [15,16]). It is shown that the KdV equation is globally well-posed for $u_0 \in H^{-1}$ [16]. It is observed that the KdV equation does not accommodate wave breaking (by wave breaking we understand that the wave remains bounded but its slope becomes unbounded in finite time [17]).

For $\alpha = 1$, $b = 2$ and $\varepsilon = 0$, Eq. (1.2) becomes the Camassa–Holm equation, modeling the unidirectional propagation of shallow water waves over a flat bottom, where $u(t, x)$ stands for the fluid velocity at time t in the spatial x direction [12,18–21]. The Camassa–Holm equation is also a model for the propagation of axially symmetric waves in hyperelastic rods [22,23]. It has a bi-Hamiltonian structure [24,25] and is completely integrable [18,26–30]. Its solitary waves are peaked [31]. The peaked solitons are orbital stable [32]. The explicit interaction of the peaked solitons is given in [33]. The peakons capture a characteristic of the traveling waves of greatest height—exact traveling solutions of the governing equations for water waves with a peak at their crest, of [34–36]. Simpler approximate shallow water models (like KdV) do not present traveling wave solutions with this feature (see [17]).

The Cauchy problems of the Camassa–Holm equation have been studied extensively. It has been shown that this equation is locally well-posed [37–41] for the initial data $u_0 \in H^s(I)$ with $s > 3/2$, where $I = \mathbb{R}$ or $I = \mathbb{R}/\mathbb{Z}$. More interestingly, it has global strong solutions [37,39,42,43] and also blow-up solutions in finite time [30,37,39,42–45]. On the other hand, it has global weak solutions in H^1 of [39,46–51]. It is observed that if u is the solution of the Camassa–Holm equation with the initial data u_0 in $H^1(\mathbb{R})$, we have for all $t > 0$,

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \sqrt{2}\,\|u(t,\cdot)\|_{H^1(\mathbb{R})} \leq \sqrt{2}\,\|u_0(\cdot)\|_{H^1(\mathbb{R})}.$$

In comparison with the KdV equation, the Camassa–Holm equation has two advantages. First, the Camassa–Holm equation represents the next order in the asymptotic expansion for shallow water waves beyond the KdV equation [2,21]. Second, the Camassa–Holm equation admits peaked traveling waves, replicating a feature that is characteristic for waves of great height—waves of largest amplitude that are exact solutions of governing equations for water waves [34–36]. Moreover, these solutions are orbital stable—that is, their shape is stable under small perturbations and therefore these waves are recognizable physically [32,52,53]. The stability holds for the Camassa–Holm equation but not for the solutions of the governing equations, as water waves are due to the small-amplitude shallow water regime in which this is the valid model of [54]. It is worth pointing out that the equation models breaking waves [31,44]. The smooth solutions of the Camassa–Holm equation have an infinite propagation speed of [55].

For $\alpha = 1$, $b = 2$ and $\varepsilon \neq 0$ in Eq. (1.2), it becomes

$$u_t - u_{xxt} - \varepsilon(u - u_{xx})_{xx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.4}$$

which is the one-dimensional version of the three dimensional Navier–Stokes-alpha model for turbulence [56,57], we call Eq. (1.4) the viscous Camassa–Holm equation.

If $\alpha = 1$, $b = 3$ and $\varepsilon = 0$ in Eq. (1.2), then we find the Degasperis–Procesi equation [5]. The formal integrability of the Degasperis–Procesi equation was obtained in [58] by constructing a Lax pair. It has a bi-Hamiltonian structure with an infinite sequence of conserved quantities, and admits exact peakon solutions which are analogous to the Camassa–Holm peakons [58].

The Degasperis–Procesi equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa–Holm shallow water equation [2,3,19,21]. Dullin etal. [2] showed that the Degasperis–Procesi equation can be obtained from the shallow water elevation equation by an appropriate Kodama transformation. An inverse scattering approach for computing n-peakon solutions to the Degasperis–Procesi equation was presented in [59]. Its traveling wave solution was investigated in [60,61]. Holm and Staley [1] studied stability of solitons and peakons numerically to the Degasperis–Procesi equation.

After the Degasperis–Procesi equation was derived, many papers were devoted to its study, of [55,59–72]. For example, the Cauchy problems for the Degasperis–Procesi equation on the line and on the circle have been studied recently. Local well-posedness of this equation has been established in [45,62] for $u_0 \in H^s(I)$ with $s > 3/2$, where $I = \mathbb{R}$ or $I = \mathbb{R}/\mathbb{Z}$. Similar to the Camassa–Holm equation, the Degasperis–Procesi equation has also global strong solutions [40,63–66] as well as finite time blow-up solutions [45,62–67]. On the other hand, it has global weak solutions in $H^1(I)$ [63,66,68].