Nonlinear Analysis 72 (2010) 1031–1043

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Weighted $L^q - L^1$ estimate of the Stokes flow in the half space

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A R T I C L E I N F O

Article history:
Received 24 March 2009
Accepted 14 July 2009

Keywords:
Half space
Stokes
Weighted

A B S T R A C T

In this paper, we derive the weighted $L^q - L^1$ estimate of the Stokes solution $\mathbf{u}$ in the half space. When $1 < q < \infty$, we obtained that

$$\||\bar{\mathbf{x}}|^r\mathbf{u}\|_{L^q} \le Ct^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}\||\bar{\mathbf{x}}|^r x_n\mathbf{a}\|_{L^1} + Ct^{\frac{r-1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}\|x_n\mathbf{a}\|_{L^1}$$

for $0 \le r < (n-1)(1-\frac{1}{q})$ and

$$\|x_n^r\mathbf{u}\|_{L^q} \le Ct^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}\|x_n^{r+1}\mathbf{a}\|_{L^1} + Ct^{\frac{r-1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}\|x_n\mathbf{a}\|_{L^1}$$

for $0 \le r < n(1-\frac{1}{q})$.

1. Introduction

Let us consider the Stokes equations in the n-dimensional half space $\mathbb{R}^n_+$, $n \ge 2$:

$$\operatorname{div}\mathbf{u} = 0, \qquad \frac{\partial \mathbf{u}}{\partial t} - \Delta\mathbf{u} + \nabla p = 0 \quad \text{in } \mathbb{R}^n_+ \times (0,\infty) \tag{1}$$

with initial condition $\mathbf{u}(\mathbf{x}, 0) = \mathbf{a}(\mathbf{x})$ in $\mathbb{R}^n_+$ and with no slip boundary condition $\mathbf{u} = 0$ on $x_n = 0$. Here $\mathbf{a}$ is a divergence free vector field with zero boundary value.

Let A be the generating operator of the Stokes semigroup so that the solution of the Stokes equation (1) is written by $\mathbf{u}(\mathbf{x}, t) = \mathrm{e}^{-tA}\mathbf{a}(\mathbf{x})$.

The $L^q - L^1$ estimate of the Stokes flow in the half space has been derived by Borchers and Miyakawa [1] via Semi-group method for $1 < q \le \infty$:

$$\|\mathrm{e}^{-tA}\mathbf{a}\|_{L^q} \le Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}\|\mathbf{a}\|_{L^1}.$$

The $L^1 - L^1$ estimate for $\nabla\mathbf{u}$ in the half space has been shown by Giga, Matsui and Shimizu [2]. Fujigaki and Miyakawa [3] derived more rapid L^q estimate of the Stokes flow with initial data in the weighted L^1 space:

$$\|\mathrm{e}^{-tA}\mathbf{a}\|_{L^q} \le Ct^{-\frac{1}{2}-\frac{n}{2}\left(1-\frac{1}{q}\right)}\|x_n\mathbf{a}\|_{L^1}.$$

Bae [4] has considered a rapid L^1 estimate with an initial data with the special condition $\int_{-\infty}^{\infty}\mathbf{a}(\mathbf{x})\mathrm{d}\mathbf{x}_i = 0$ for some $i = 1, \ldots, n-1$, and in this case

$$\|\mathrm{e}^{-tA}\mathbf{a}\|_{L^1} \le Ct^{-\frac{1}{2}}\|x_n\mathbf{a}\|_{L^1}.$$

The solvability of the Navier–Stokes equations and decay rate properties of the solution in the weighted L^q spaces has been considered by many mathematicians.

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0362-546X/$ – see front matter
doi:10.1016/j.na.2009.07.044

There is abundant literature for the Cauchy problem related to the weighted spaces and, the existing results are almost optimal (see [5–13], etc.).

On the other hand, there is less literature for the exterior domain problem and half space problem related to the weighted spaces and much improvement is required (see [14–18,3,19]).

The weighted estimate of the Stokes flow is the first step for the study of the solvability and the asymptotic properties of Navier–Stokes flow in the weighted L^q spaces. In this paper, we derive the weighted L^q estimate of the Stokes flow for $1 < q < \infty$. We hope this result could be applied to the solvability of the Navier–Stokes equations in a weighted $L^q(\mathbb{R}^n_+)$ spaces.

The following are the statements of the main theorems.

Theorem 1.1. *Let $x_n(1+|\mathbf{x}|)^r\mathbf{a} \in L^1$. Let $\mathbf{u}$ be the solution of the Stokes equations (1) with initial data $\mathbf{a}$.*

Let us consider $1 < q < \infty$. Then we have that

$$\||\bar{\cdot}|^r\mathbf{u}\|_{L^q} \leq Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{1}{2}}\||\bar{\cdot}|^r x_n\mathbf{a}\|_{L^1} + Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)+\frac{r-1}{2}}\|x_n\mathbf{a}\|_{L^1}$$

for $0 \leq r < (n-1)(1-\frac{1}{q})$ and

$$\|x_n^r\mathbf{u}\|_{L^q} \leq Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)-\frac{1}{2}}\|x_n^{r+1}\mathbf{a}\|_{L^1} + Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)+\frac{r-1}{2}}\|x_n\mathbf{a}\|_{L^1}$$

for $0 \leq r < n(1-\frac{1}{q})$.

Our result can be compared with the result of Bae [20], where weighted $L^q - L^1$ estimate of Stokes flow have been considered for $1 < q < \infty$ with weight function $|\mathbf{x}|^r$ for $0 \leq r < n\left(1-\frac{1}{q}\right)$, $|\bar{\mathbf{x}}|^r$ for $0 \leq r < (n-1)(1-\frac{1}{q})$ and x_n^r for $0 \leq r < 1-\frac{1}{q}$.

In [20] it is claimed that UKai's operator $\bar{R}\cdot S(\bar{R}\cdot S + R_n)$ is L^q bounded with the weight function $|\mathbf{x}|^r$ for $0 < r < n(1-\frac{1}{q})$. Indeed Ukai's operator $\bar{R}\cdot S(\bar{R}\cdot S + R_n)$ is L^q bounded without any weight. However, we are not sure if the operator is L^q bounded with the weight function $|\mathbf{x}|^r$ for $0 < r < n(1-\frac{1}{q})$. (It is a well-known theory that a Calderon–Zygmund operator is L^q bounded with the weight functions $|\mathbf{x}|^r$ for $0 < r < n(1-\frac{1}{q})$, $|\bar{\mathbf{x}}|^r$ for $0 < r < (n-1)(1-\frac{1}{q})$ and x_n^r for $0 < r < 1-\frac{1}{q}$. However in [20] it is not shown whether Ukai's operator is Calderon–Zygmund type or not.)

When a rapid decaying initial data is concerned, there is a paper of Crispo and Maremonti. In [16], they derived a weighted $L^\infty - L^\infty$ estimate of Stokes flow and making use of it, they proved the existence of rapidly decaying Navier–Stokes flow. (They also considered the bounded estimate of the Stokes flow with continuous, bounded but nonconvergent initial data in [21], and making use of it, they proved the existence of bounded and continuous Navier–Stokes flow in [22].)

We emphasize that in Theorem 1.1 the L^q bound with the weight function x_n^r for $0 \leq r < n(1-\frac{1}{q})$ is an improved result compared to [20]. For the purpose of it, the following Lemma plays an important role whose proof is given in Section 3.6.

Lemma 1.2. *Let $T = R_iR_j$, $i, j = 1, \ldots, n$. Let $0 \leq r < n(1-\frac{1}{q})$. Let k be a nonnegative integer and $0 \leq \alpha < 1$ with $r = k+\alpha$. Then we have that*

$$\|x_n^r Tf\|_{L^q(\mathbb{R}^n_+)} \leq C\sum_{l\leq k}\left(\||x_n|^\alpha\nabla^l(x_n^l f)\|_{L^{\frac{np}{n+kp}}} + \|\nabla^l(x_n^l f)\|_{L^{\frac{np}{n+rp}}}\right).$$

In [16,22,21], the pressure estimates of the Stokes flow and Navier–Stokes flow have also been considered for rapidly decaying initial data [16], and for bounded or increasing initial data [22,21]. In those papers, Solonnikov's solution formula [23,24] has been used. In this paper we make use of Ukai's solution formula [25]. We leave the pressure estimate in weighted L^q spaces for the future.

2. Ukai's solution formula

Let Γ be the n-dimensional Gaussian kernel defined by $\Gamma(\mathbf{x},t) = \Gamma_t(\mathbf{x}) = \frac{1}{(2\pi t)^{\frac{n}{2}}}\mathrm{e}^{-\frac{|\mathbf{x}|^2}{4t}}$. Define the operators $\mathrm{e}^{-t\Delta}$ and e^{-tB} by

$$\mathrm{e}^{-t\Delta}a(\mathbf{x}) = \int_{\mathbb{R}^n}\Gamma_t(\mathbf{x}-\mathbf{y})a(\mathbf{y})\mathrm{d}y \quad \text{on } \mathbb{R}^n.$$

and

$$\mathrm{e}^{-tB}a(\mathbf{x}) = \int_{\mathbb{R}^n_+}[\Gamma_t(\mathbf{x}-\mathbf{y}) - \Gamma_t(\mathbf{x}-\mathbf{y}^*)]a(\mathbf{y})\mathrm{d}y,$$

respectively. Here $\mathbf{y}^* = (\bar{\mathbf{y}}, -y_n)$ for $\mathbf{y} = (\bar{\mathbf{y}}, y_n)$.