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Properties of non-simultaneous blow-up solutions in nonlocal parabolic equations

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ABSTRACT

This paper deals with blow-up solutions in parabolic equations coupled via nonlocal nonlinearities, subject to homogeneous Dirichlet conditions. Firstly, some criteria on non-simultaneous and simultaneous blow-up are given, including four kinds of phenomena: (i) the existence of non-simultaneous blow-up; (ii) the coexistence of non-simultaneous and simultaneous blow-up; (iii) any blow-up must be simultaneous; (iv) any blow-up must be non-simultaneous. Next, total versus single point blow-up are classified completely. Moreover, blow-up rates are obtained for both non-simultaneous and simultaneous blow-up solutions.

1. Introduction

In this paper, we consider the parabolic equations coupled via nonlocal nonlinearities

$$\begin{cases} u_t = \Delta u + u^m \displaystyle\int_\Omega v^n(y,t)\mathrm{d}y, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v + v^q \displaystyle\int_\Omega u^p(y,t)\mathrm{d}y, & (x,t) \in \Omega \times (0,T), \\ u = v = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x),\, v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = B_R = \{|x| < R\} \subset \mathbb{R}^N$; exponents $m, n, p, q \geq 1$; T represents the maximal existence time of the solutions; radially symmetric u_0, v_0 are nonnegative, nontrivial, radially decreasing functions, vanishing on ∂B_R. The existence and uniqueness of local classical solutions to (1.1) is well known (see, for example, [1,2]). Nonlinear parabolic equations coupled via nonlocal sources, like (1.1), come from population dynamics, chemical reactions, heat transfer, etc., where u and v represent the densities of two biological populations during a migration, the thickness of two kinds of chemical reactants, the temperatures of two different materials during a propagation, etc.

Chen [3] discussed the following parabolic equations coupled with local nonlinearities

$$u_t = \Delta u + u^m v^n, \qquad v_t = \Delta v + u^p v^q, \quad (x,t) \in \Omega \times (0,T), \tag{1.2}$$

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subject to homogeneous Dirichlet boundary conditions with $p > m-1$ and $n > q-1$, where Ω is a general bounded domain of $\mathbb{R}^N$. He proved that, if $m, q \le 1$ and $pn \le (1-m)(1-q)$, then all solutions are global, while both global and blow-up solutions coexist if $m > 1$, or $q > 1$, or $pn > (1-m)(1-q)$, where the blow-up solutions are defined as

$$\limsup_{t\to T} (\|u(\cdot,t)\|_\infty + \|v(\cdot,t)\|_\infty) = +\infty. \tag{1.3}$$

The non-simultaneous blow-up had been observed and discussed by Quirós and Rossi [4] for the Cauchy problem of (1.2) in $\mathbb{R}^N$, where simultaneous blow-up means that

$$\limsup_{t\to T} \|u(\cdot,t)\|_\infty = \limsup_{t\to T} \|v(\cdot,t)\|_\infty = +\infty,$$

and non-simultaneous blow-up is, e.g., $\sup_{t\in[0,T)} \|v(\cdot,t)\|_\infty < +\infty = \limsup_{t\to T} \|u(\cdot,t)\|_\infty$. They proved that there exists initial data such that u blows up while v remains bounded if $m > p+1$; If u blows up at some point $x_0 \in \mathbb{R}^N$ while v remains bounded and

$$u(x,t) \ge c(T-t)^{-\frac{1}{m-1}}, \qquad |x-x_0| \le K\sqrt{T-t}, \tag{1.4}$$

then $m > p+1$. It was noted that the results of [4] also hold for system (1.2). The restriction condition (1.4) had been removed by Brändle, Quirós, and Rossi [5] for the Cauchy problem of (1.2) with $N = 1$. There are also some results for local nonlinearities with respect to non-simultaneous blow-up (see, for example, [6–8]). The simultaneous blow-up rate of (1.2) was obtained by Wang [9], Zheng [10], respectively, for radially symmetric solutions in B_R as follows,

$$\max_{\bar{B}_R} u(\cdot,t) \sim (T-t)^{-\frac{n+1-q}{pn-(1-m)(1-q)}}, \qquad \max_{\bar{B}_R} v(\cdot,t) \sim (T-t)^{-\frac{p+1-m}{pn-(1-m)(1-q)}}.$$

The notation $f \sim g$ means that there exist some positive constants c and C such that $cg \le f \le Cg$. The other studies about system (1.2) were considered in [1,11–13], etc., where the blow-up criteria, blow-up rate, and blow-up profile were considered.

Li and Wang [14] considered the following localized parabolic equations

$$u_t = \Delta u + u^m(x,t)v^n(x_0,t), \qquad v_t = \Delta v + u^p(x_0,t)v^q(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.5}$$

subject to homogeneous Dirichlet boundary conditions with $m, n, p, q \ge 0$, $m + p > 0$, $n + q > 0$. They obtain that

- $m, q \le 1$: the blow-up classical solution (u, v) is simultaneous, and possesses the total blow-up pattern. Moreover, the uniform blow-up profiles are obtained.
- $m, q > 1$ with $\Omega = B_R = \{|x| < R\}$ and $x_0 = 0$: under the assumptions
  (H1) $u_0, v_0 : \bar{B}_R \to \mathbb{R}^1$ are nonnegative nontrivial, radially symmetric non-increasing continuous functions and vanish on ∂B_R;
  (H2) $\Delta u_0(x) + u_0^m(x)v_0^n(0) \ge 0$, $\Delta v_0(x) + u_0^p(0)v_0^q(x) \ge 0$, $x \in B_R$,
  the following results are obtained.
  - A sufficient condition for only simultaneous blow-up: if $p \ge m-1 > 0$ and $n \ge q-1 > 0$, then any blow-up must be simultaneous.
  - A necessary condition for the existence of simultaneous blow-up: if simultaneous blow-up happens, then the exponent regions must be $p \ge m-1 > 0$ and $n \ge q-1 > 0$ or $p < m-1$ and $n < q-1$. In addition, the simultaneous blow-up rates are obtained. (u, v) blows up only at $x = 0$.
  - The blow-up rates in space are evaluated as $u(r,t) \le Cr^{-\alpha}$, $v(r,t) \le Cr^{-\beta}$, $(r,t) \in (0,R] \times [0,T)$ with $\alpha > 2/(m-1)$ and $\beta > 2/(q-1)$, under assumptions $u_0'(r) \le -cr$, $v_0'(r) \le -cr$ in $[0, R]$.

Li, Huang, and Xie [15] considered the following nonlocal parabolic equations

$$\begin{cases} u_t = \Delta u + \int_\Omega u^m(y,t)v^n(y,t)\mathrm{d}y, & (x,t) \in \Omega \times (0,T), \\ v_t = \Delta v + \int_\Omega u^p(y,t)v^q(y,t)\mathrm{d}y, & (x,t) \in \Omega \times (0,T), \end{cases} \tag{1.6}$$

subject to homogeneous Dirichlet boundary conditions. They obtained that the solutions of (1.6) blow up under large initial data in terms of (1.3) if $m > 1$, or $q > 1$, or $pn > (1-m)(1-q)$, and they also obtained blow-up rates of solutions. There have been many other results for parabolic equations with nonlocal nonlinearities. We refer the readers to [16,14,17,2,18] and the references therein.

To our knowledge, the non-simultaneous and simultaneous blow-up, blow-up rates and blow-up sets of system (1.1) have been rarely considered before. Motivated by the above works, in the present paper, we give some criteria of non-simultaneous and simultaneous blow-up solutions in Section 2. In Section 3, blow-up sets and blow-up rates are discussed for both non-simultaneous and simultaneous blow-up solutions.