Nonlinear Analysis 73 (2010) 835–839

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Compact embeddings and proper mappings

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ARTICLE INFO

Article history:
Received 16 July 2009
Accepted 24 March 2010

MSC:
primary 58C30
secondary 65H10

Keywords:
Fixed point
Continuation methods
Surjective Implicit Function Theorem
C^1-homotopy
Proper mapping
Compact embedding
Coercive mapping
Zorn's lemma

ABSTRACT

Sufficient conditions are given to assert that a C^1-proper mapping defined on a compact embedding between Banach spaces over $\mathbb{K}$ has a fixed point. The proof of the result is based upon continuation methods.

1. Preliminaries

Let X and Y be two normed spaces. If $u : X \to Y$ is a continuous mapping, then one way of solving the equation

$$u(x) = 0 \tag{1}$$

is to embed (1) in a continuum of problems

$$H(t, x) = 0\ (0 \leq t \leq 1), \tag{2}$$

which can easily be resolved when $t = 0$. When $t = 1$, the problem (2) becomes (1). In the case when it is possible to continue the solution for all t in $[0, 1]$ then (1) is solved. This method is called continuation with respect to a parameter [1–24].

In this paper, sufficient conditions are given for proving that a C^1-proper mapping $f : X \to Y$ has a fixed point, where X, Y are Banach spaces, and $''X \subseteq Y''$ represents a compact embedding. Other conditions, sufficient for guaranteeing the existence of fixed points, have been given by the author for finite-dimensional settings [7–14,16,17,19,23,24], for infinite-dimensional settings for Fredholm mappings [15,20], for infinite-dimensional Hilbert spaces for continuous embeddings [25] and for compact Banach manifolds modelled on $\mathbb{R}^n$ [22]. Continuation methods were used both in the proofs of these papers and to prove the existence of open trajectories and lower bounds of stable stationary trajectories of ordinary differential equations [18,21]. Continuation methods are also used here. The proof supplies the existence of implicitly defined mappings whose ranges reach fixed points [1–24]. The key is the use of the Surjective Implicit Function Theorem [26], the properties of compact embeddings (see [27,26]), and Zorn's lemma and compactness (see [28]).

We briefly recall some theorems and concepts to be used.

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doi:10.1016/j.na.2010.03.037

Definitions ([27,26]). Henceforth we will assume that X, Y and Z are normed spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and $U(x, y)$ is an open neighbourhood of the point (x, y).

Mapping $F : X \to Y$ is called *weakly coercive* if and only if $\|F(x)\| \to \infty$ as $\|x\| \to \infty$.

Mapping $F : D(F) \subseteq X \to Y$ is said to be *compact* whenever it is continuous and the image $F(B)$ is relatively compact (i.e. its closure $\overline{F(B)}$ is compact in Y) for every bounded subset $B \subset D(F)$). Obviously, this second property is equivalent to the following: If (u_n) is a bounded sequence in $D(F)$, then there exists a subsequence $(u_{n'})$ such that the subsequence $(F(u_{n'}))$ is convergent in Y.

Mapping F is said to be *proper* whenever the pre-image $F^{-1}(K)$ of every compact subset $K \subset Y$ is also a compact subset of $D(F)$.

Let X and Y be Banach spaces over $\mathbb{K}$, with $X \subseteq Y$. The *embedding operator* $i : X \to Y$ is defined by $i(u) = u$ for all $u \in X$.

The *embedding* $X \subseteq Y$ is called *continuous* if i is continuous, i.e., $\| i(u) \|_Y \leq \text{const} \| u \|_X$ for all $u \in X$.

The *embedding* $X \subseteq Y$ is called *compact* if i is compact, i.e., $\| i(u) \|_Y \leq \text{const} \| u \|_X$ for all $u \in X$, and each bounded sequence (u_n) in X has a subsequence $(u_{n'})$ such that the sequence $(i(u_{n'}))$ is convergent in Y.

More generally, we speak of an *embedding* if there are two Banach spaces X and Y over $\mathbb{K}$ and an injective linear operator $i : X \to Y$. Since i is injective, u can be identified with $i(u)$. This can be represented as "$X \subseteq Y$".

The expression $H_x(x, t)$ denotes the partial F-derivative of H with respect to X at the point (x, t), where $H : U(x, t) \subseteq X \times Y \to Z$, and where $U(x, t)$ is an open neighbourhood of the point (x, t).

The expression $f'(x)$ denotes the F-derivative of f at the point x, where $f : X \to Y$.

Let $\mathcal{L}(X, Y)$ denote the set of all linear continuous mappings $L : X \to Y$.

Theorem 1. *The Surjective Implicit Function Theorem. ([26], pp.268–269). Let X, Y, Z be Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let*

$$F : U(u_0, v_0) \subseteq X \times Y \to Z$$

be a C^1-mapping on an open neighbourhood of the point (u_0, v_0). Suppose that:

(i) $F(u_0, v_0) = 0$, *and*
(ii) $F_v(u_0, v_0) : Y \to Z$ *is surjective.*

Then the following are true:

(a) *Let $r > 0$. There is a number $\rho > 0$ such that, for each given $u \in X$ with $\| u - u_0 \| < \rho$, the equation*

$$F(u, v) = 0$$

has a solution v, denoted by $v(u)$, such that $\| v - v_0 \| < r$. In particular, the limit $u \to u_0$ in X implies $v(u) \to v_0$.
(b) *There is a number $d > 0$ such that $\| v(u) \| \leq d \| F_v(u_0, v_0)v(u) \|$.*

2. Compact embeddings and proper mappings

Theorem 2. *Let $''X \subseteq Y''$ be a compact embedding, where X $(X, \| \cdot \|_X)$ and Y $(Y, \| \cdot \|_Y)$ are Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and*

$$i : X \to Y$$

is the injective linear continuous mapping. Let us suppose that the following hold:

(i) *$g : X \to Y$ is a C^1-proper mapping and mapping $ti(x) - g(x)$ is weakly coercive for each $t \in [0, 1]$.*
(ii) *$\forall (x, t) \in X \times [0, 1]$ such that $ti(x) - g(x) = 0$, the mapping $ti(x) - g'(x) \in \mathcal{L}(X, Y)$ is surjective.*
(iii) *There is an x_0 such that $g(x_0) = 0$.*

Then the following hold:

(a) *There is an $x^* \in X$ such that $g(x^*) = i(x^*)$.*
(b) *If $X \subseteq Y$ and $i(u) = u$, then g has a fixed point x^*, i.e., $g(x^*) = x^*$.*

Proof. (a). Let us construct the mapping

$$H : X \times [0, 1] \to Y, \quad \text{where } H(x, t) := ti(x) - g(x).$$

We will prove here that H is a proper mapping, and since $0 \in Y$ is a compact set then $H^{-1}(0)$ is also a compact set.

Let C be any fixed compact subset of Y, and let any sequence such as $(H(x_n, t_n))_{n\geq 1}$ which belongs to C be fixed. It suffices to show that the sequence $((x_n, t_n))_{n\geq 1}$ contains a convergent subsequence $((x_{n''}, t_{n''}))_{n''\geq 1}$. Since

$$(x_{n''}, t_{n''}) \to (u, t) \quad \text{as } n'' \to \infty,$$

and H is continuous, then $H(u, t) \in C$, that is, $(u, t) \in H^{-1}(C)$, and therefore $H^{-1}(C)$ is a compact set and H a proper mapping.