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On the $\mu_{M,D}$-orthogonal exponentials

Jian-Lin Li

College of Mathematics and Information Science, Shaanxi Normal University, Xi'an 710062, PR China

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ABSTRACT

Let $\mu_{M,D}$ be a self-affine measure corresponding to a given affine iterated function system $\{\phi_d(x) = M^{-1}(x+d)\}_{d\in D}$. In the present paper we will study the problem of how to determine the $L^2(\mu_{M,D})$-space has finite or infinite orthogonal exponentials. Such research is motivated by a conjecture on the non-spectrality of $\mu_{M,D}$. We first obtain a non-spectral criterion which extends the result of Dutkay and Jorgensen. In opposition to the condition of this criterion, we then obtain some conditions which imply the infinite orthogonal systems in $L^2(\mu_{M,D})$. These are necessary for further investigation on the spectrality of $\mu_{M,D}$. As an application, we completely settle the corresponding problem for the generalized planar Sierpinski family.

1. Introduction

Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix, that is, all the eigenvalues of the integer matrix M have modulus greater than 1. Associated with a finite subset $D \subset \mathbb{Z}^n$, there exists a unique nonempty compact set $T := T(M, D)$ such that $MT = \bigcup_{d\in D}(T+d)$. More precisely, $T(M, D)$ is an attractor (or invariant set) of the affine iterated function system (IFS) $\{\phi_d(x) = M^{-1}(x+d)\}_{d\in D}$. Denote by $|D|$ the cardinality of D. Relating to the IFS $\{\phi_d\}_{d\in D}$, there also exists a unique probability measure $\mu := \mu_{M,D}$ satisfying

$$\mu = \frac{1}{|D|}\sum_{d\in D}\mu\circ\phi_d^{-1}. \tag{1.1}$$

In general, the invariant set $T(M, D)$ behaves like a fractal, highly non-linear, it includes complicated geometries. The invariant measure $\mu_{M,D}$, which is also called *self-affine measure*, includes the restriction of n-dimensional Lebesgue measure μ_L. Moreover $\mu_{M,D}$ is supported on $T(M, D)$ (cf. [1]).

It is well known that the orthogonal exponentials in $L^2(\mu_L)$ play a central role in Fourier analysis on Euclidean space. What happens to the Fourier basis if we attempt to replace μ_L by more general measures such as $\mu_{M,D}$? There are a wide range of interests in this question after the work of Jorgensen and Pedersen [2]. In this paper we will characterize the orthogonal exponentials in $L^2(\mu_{M,D})$.

Recall that for a probability measure μ of compact support on $\mathbb{R}^n$, we call μ a *spectral measure* if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that $E(\Lambda) := \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis (Fourier basis) for $L^2(\mu)$. The set Λ is then called a *spectrum* for μ; we also say that (μ, Λ) is a *spectral pair* (cf. [3]). Spectral measure is a generalization of the *spectral set* introduced by Fuglede [4]. In recent years, the research on the spectrality or non-spectrality of a self-affine measure $\mu_{M,D}$ has received much attention (see e.g., [5–11] and references cited therein). The analysis of spectral properties of the self-affine measure $\mu_{M,D}$ is motivated by and has influence on a variety of areas outside the fractal theory. While there are many efforts at understanding the spectrality of $\mu_{M,D}$, the research here will mainly concentrate on the problem of how to

E-mail address: jllimath@yahoo.com.cn.

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determine whether the $L^2(\mu_{M,D})$-space has finite or infinite orthogonal exponentials. This is a fundamental problem which consists in determining conditions on M and D such that $\mu_{M,D}$-orthogonal exponentials are finite or infinite. It has very close connections with the non-spectral problem on $\mu_{M,D}$.

The non-spectral problem on the self-affine measure $\mu_{M,D}$ consists of two classes (cf. [6,10]). The first class is:
Class (I). There are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, that is, $\mu_{M,D}$-orthogonal exponentials contain at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them.

The corresponding conjecture is

Conjecture 1. *For an expanding integer matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$, denote by $W(m)$ the non-negative integer combination of the prime divisors of $m := |\det(M)|$. If $|D| \notin W(m)$, then $\mu_{M,D}$ is a non-spectral measure and the non-spectral problem on this $\mu_{M,D}$ falls into class* (I).

Recently, the author [10] proved Conjecture 1 for a class of planar self-affine measures with a three-element digit set (the generalized planar Sierpinski family). Since conjecture 1 is open for a given pair (M, D), the first problem we meet in expanding the classical Fourier analysis to measure $\mu_{M,D}$ is still the above-mentioned problem. That is, how to determine whether the $L^2(\mu_{M,D})$-space has finite or infinite orthogonal exponentials. There are some results in this direction, such as the compatible pair condition, which implies the infiniteness of orthogonal exponentials in $L^2(\mu_{M,D})$-space (cf. [2,12–14]), and Theorem 3.1 in [6], which gives a criterion of non-spectrality on the finiteness of orthogonal exponentials in $L^2(\mu_{M,D})$-space, but we are still far from settling this problem. The previous studies on this interesting topic also leave several related problems which have motivated the present research.

In the present paper, we will further study the above-mentioned problem in detail. We first obtain a criterion of non-spectrality in Section 2, which extends the corresponding Theorem 3.1 in [6]. In opposition to the condition of this criterion, we then discuss the conditions that imply the infinite orthogonal systems in Section 3. We obtain some easy-check conditions which imply the infinite orthogonal systems. These are necessary for further investigation on the spectrality of $\mu_{M,D}$. As an application of the above-established results, in Section 4 we solve the problem of determining the finiteness or infiniteness of orthogonal exponentials on the generalized planar Sierpinski family. Consequently, the non-spectral problem of class (I) on the Sierpinski family is completely settled. For this interesting family, the conclusion of the above Conjecture 1 holds even in certain cases when $|D| \in W(m)$. In the final section, we give general forms of the established results.

2. A criterion of non-spectrality

For an expanding matrix $M \in M_n(\mathbb{Z})$ and a finite digit set $D \subset \mathbb{Z}^n$, the Fourier transform of the self-affine measure $\mu_{M,D}$ is

$$\hat{\mu}_{M,D}(\xi) := \int e^{2\pi i\langle x,\xi\rangle} d\mu_{M,D}(x) = \prod_{j=1}^{\infty} m_D(M^{*-j}\xi), \quad (\xi \in \mathbb{R}^n) \tag{2.1}$$

where

$$m_D(x) := \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle d,x\rangle}, \quad (x \in \mathbb{R}^n) \tag{2.2}$$

and M^* denotes the transposed conjugate of M, in fact $M^* = M^t$. For any $\lambda_1, \lambda_2 \in \mathbb{R}^n$, $\lambda_1 \neq \lambda_2$, the orthogonality condition

$$\langle e^{2\pi i\langle\lambda_1,x\rangle}, e^{2\pi i\langle\lambda_2,x\rangle}\rangle_{L^2(\mu_{M,D})} = \int e^{2\pi i\langle\lambda_1-\lambda_2,x\rangle} d\mu_{M,D} = \hat{\mu}_{M,D}(\lambda_1 - \lambda_2) = 0 \tag{2.3}$$

directly relates to the zero set $Z(\hat{\mu}_{M,D})$ of $\hat{\mu}_{M,D}$. From (2.1), we obtain that

$$Z(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty}\{\xi \in \mathbb{R}^n : m_D(M^{*-j}\xi) = 0\}. \tag{2.4}$$

It was first observed by Jorgensen and Pedersen [2] that certain IFS fractals have Fourier bases. And if there are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$, then $\mu_{M,D}$ is non-spectrality. Usually, it is not easy to prove the finiteness of orthogonal exponentials in a given space $L^2(\mu_{M,D})$, because it involves an intrinsic arithmetic of the finite set of functions making up the IFS $\{\phi_d(x)\}_{d\in D}$. In this regard, Dutkay and Jorgensen [6, Theorem 3.1 (i)] established the following criterion of non-spectrality for certain self-affine measures.

Theorem A. *Let Z be the set of the zeros of function $m_D(x)$ in $[0, 1)^n$. If Z is contained in a set $Z' \subset [0, 1)^n$ of finite cardinality $|Z'|$ that satisfies the properties:*

$$M^*(Z' + \mathbb{Z}^n) \subseteq Z' + \mathbb{Z}^n \quad \text{and} \quad 0 \notin Z', \tag{2.5}$$

then there exist at most $|Z'| + 1$ mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. In particular, $\mu_{M,D}$ is not a spectral measure.