Nonlinear Analysis 73 (2010) 979–986

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On the existence of ground states for nonlinear Schrödinger–Poisson equation

Mariano De Leo *

Instituto de Ciencias, Universidad Nacional de General Sarmiento, J.M. Gutiérrez 1150 (1613) Los Polvorines, Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 7 August 2008
Accepted 22 April 2010

MSC:
35Q55
37K05
35K55

Keywords:
Nonlinear Schrödinger–Poisson
Ground states
Doping profile
Conservation laws

ABSTRACT

This paper is concerned with the existence of ground states for the Schrödinger–Poisson equation $i\,\partial_t u = -\partial_x^2 u + V(u)\,u - f(|u|^2)\,u$, where $V(u)$ is a Hartree type nonlinearity, stemming from the coupling with the Poisson equation, which includes the so-called doping profile or *impurities*. By means of variational methods in the energy space we show that ground states exist and belong to the Schwartz space of rapidly decreasing functions whenever total charge not exceed some critical value, it is also shown that for values of the total charge greater than this critical value, energy is not bounded from below. In addition, we show that this critical value is the total charge given by the impurities.

1. Introduction

In this article we are concerned with the existence of ground states of the following 1D Schrödinger–Poisson problem

$$\mathrm{i}\partial_t u = -\partial_x^2 u + \frac{1}{2}\left(|x| * \left(\mathcal{C} - |u|^2\right)\right) u - f(|u|^2)u, \quad x \in \mathbb{R}. \tag{1.1}$$

Here, $\mathcal{C}$ denotes the fixed positively charged background ions or *impurities* (in what follows, will be referred to as *the doping profile*) and it is assumed to be a (positive) regular function with compact support; the non-local term takes into account the coupling with the Poisson equation; and the term $f(|u|^2)u$ represents a local interaction (which is intended to take into account the Pauli exclusion principle for fermions). For further details in the (semiconductors) model problem, from both of physical or mathematical nature, see [1] and the references therein.

The existence of solitons is currently regarded as one of the most interesting research topics in modern nonlinear wave theory [2–4]. Actually, in the nonlinear Schrödinger equation $\mathrm{i}\partial_t u = -\partial_x^2 u + g(u)$, the global existence has been widely studied from the earlier works of Strauss [5], Ginibre–Velo [6], Berestycki–Lions [7], Weinstein [8] – mainly concerned with local interactions – including the later works of Tsutsumi [9], Ogawa–Tsutsumi [10], to the recent works of Wei-Chen [11] and Borgna [12]. However, in low dimensions the coupling with the Poisson equation appears as a delicate problem mainly due to the treatment of low frequencies; actually, the existence of global solutions in such a case was first given by Steinrück [13] who only considered the related Hartree type potential, then by Stimming [14] who adapted the proof given by the former in order to include the exchange potential, and later on by De Leo–Rial [1] extended the result to the semiconductor model including the doping profile into the Poisson equation (and also improved the decay at infinity requirements).

* Tel.: +54 11 4469 7500; fax: +54 11 4469 7506.
E-mail addresses: mdeleo@ungs.edu.ar, marianodeleo@gmail.com.

0362-546X/$ – see front matter
doi:10.1016/j.na.2010.04.025

We finally recall that ground states of the system are, by definition (see, for instance, [7]), minimizers of the energy functional subject to constraints on conserved functionals associated with symmetries of the system; in addition, for physical reasons, ground states are sometimes asked to be positive real functions. On the other hand, any solution of (1.1) (i.e., a standing wave) of the form $u(x,t) = e^{-i\lambda t}\phi(x)$ leads to a solution ϕ of the following nonlinear Euclidean scalar field equation:

$$L(\phi) := -\partial_x^2\phi + \frac{1}{2}\left(|x| * \left(\mathcal{C} - |\phi|^2\right)\right)\phi - f(|\phi|^2)\phi = \lambda\phi, \tag{1.2}$$

where λ is a real constant.

This work is organized as follows: in Section 2 we fix some notation and give a list of useful results. In Section 3 we show that the constrained variational problem related to the energy yields a solution of the problem given by (1.2) while Section 4 is devoted to the existence of minimizers in the working space and its regularity:

Theorems 4.1 and 4.3. *Let $S_b = \{\phi \in \mathcal{H} : \|\phi\|_{L^2}^2 = b\}$. Assume that $b \le \|\mathcal{C}\|_{L^1}$. Then there exists a positive real function ϕ_b such that $E(\phi_b) = \min\{E(\phi) : \phi \in S_b\}$.*

Theorem 4.2. *Let ϕ_b be the ground state given by Theorem 4.1 (subcritical case). Then ϕ_b belongs to the Schwartz space of rapidly decreasing functions.*

2. Preliminaries

Notation. Throughout this article we will use the following notation.

- $H^1 := \{\varphi \in L^2 : \int (1+|k|^2)^{1/2}|\mathcal{F}(\varphi)|^2 < \infty\}$. Where $\mathcal{F}(\cdot)$ represents the Fourier transform.
- $\mathcal{S} := \{\varphi \in C^\infty : \forall\, 0 \le r, s,\ \sup |x|^r|\partial_x^s\varphi| < +\infty\}$ the Schwartz space of rapidly decreasing functions.
- $L_\mu^2 := \{\varphi \in L^2 : \int |\varphi(x)|^2\mu(x)^2 < \infty\}$, where $\mu(x) := (1+x^2)^{1/4}$.
- $\mathcal{H} := H^1 \cap L_\mu^2$.
- $\|\cdot\|_{\mathcal{H}}^2 := \|\cdot\|_{H^1}^2 + \|\cdot\|_{L_\mu^2}^2$, norm in $\mathcal{H}$.
- $\|\cdot\|_X$ norm in X, a Banach space different from $\mathcal{H}$.

Results. We will make use of the following lemmas, mainly concerned with compactness properties.

Lemma 2.1. *Let $1 \le p < \infty$. Let $\{f_j\}j \in J \subseteq L^p(\mathbb{R})$ be a bounded sequence such that*

- *$\forall\varepsilon > 0$, there exists $R > 0$ such that $\forall j \in J$, $\int_{|x|>R} |f_j|^p < \varepsilon^p$. Mass is uniformly located.*
- *$\forall\varepsilon > 0$ there exists $\delta > 0$ such that $\forall |h| < \delta$, $\int |f_j(x+h) - f_j(x)|^p dx < \varepsilon^p$. Prevents concentration.*

Then, there exist $f \in L^p$ and a sequence f_n such that $f_n \xrightarrow{L^p} f$.

Proof. See [15], Theorem 2.32. □

Lemma 2.2. *The imbedding $\mathcal{H} \hookrightarrow L^2$ is compact.*

Proof. Let $\{g_n\}_{n\in\mathbb{N}} \subseteq \mathcal{H}$, be a bounded sequence, and take $M > 0$ such that $\|g_n\|_{\mathcal{H}} \le M$. Since, $\int_{|x|>R} |g_n|^2 \le R^{-1} \int_{|x|>R} |x||g_n|^2 \le R^{-1}\|g_n\|_{L_\mu^2}^2 \le R^{-1}\|g_n\|_{\mathcal{H}}^2 \le R^{-1}M^2$, the mass is uniformly located. On the other hand, the inequality $\int_I |g_n(x+h) - g_n(x)|^2 \le Ch^2\|g_n\|_{H^1}^2 \le Ch^2M^2$, prevents concentration. Therefore, the proof follows from Lemma 2.1. □

Lemma 2.3. *Let $\phi_n \in L^2$ be such that $\|\phi_n\|_{L^2} = 1$. Then it only may occur one of the following situations.*

(a) *There exist a number $R_0 \in \mathbb{R}$, a sequence $R_n \to \infty$, and two subsequences $\phi_n^{(k)} \in L^2$, $k = 1, 2$, such that $\phi_n = \phi_n^{(1)} + \phi_n^{(2)}$, where $\phi_n^{(k)}$ verifies,*
   - $\mathrm{Supp}(\phi_n^{(1)}) \subseteq [-R_0, R_0]$, $\|\phi_n^{(1)}\|_{L^2} = 1 - \gamma$
   - $\mathrm{Supp}(\phi_n^{(2)}) \subseteq \mathbb{R} - [-R_n, R_n]$, $\|\phi_n^{(2)}\|_{L^2} = \gamma$,

   and $0 < \gamma < 1$.

(b) *For each I interval such that $|I| < \infty$ and each $\varepsilon > 0$ there exists n_0 such that for $n > n_0$*

$$\int_I |\phi_n|^2 < \varepsilon.$$

(c) *For each $\varepsilon > 0$ there exists $R = R(\varepsilon) > 0$ such that*

$$\int_{[-R,R]} |\phi|^2 \ge 1 - \varepsilon.$$

Proof. See [16], Lemma I,1. □