Nonlinear Analysis 73 (2010) 1043–1056

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Scattering for the focusing $\dot{H}^{1/2}$-critical Hartree equation in energy space

Yanfang Gao [a,*], Haigen Wu [b,c]

[a] Institute of Mathematics, Jilin University, Changchun, 130012, China
[b] Institute of Systems Science, Chinese Academy of Science, Beijing 100190, China
[c] School of Mathematics, Henan Polytechnic University, Jiaozuo, 454000, China

ARTICLE INFO

Article history:
Received 5 February 2010
Accepted 22 April 2010

MSC:
primary 35Q55

Keywords:
Hartree equation
Scattering
Profile decomposition
Blow-up

ABSTRACT

We investigate the focusing nonlinear Schrödinger equation (NLS) of Hartree type $\mathrm{i}\partial_t u + \Delta u = -(|\cdot|^{2-d} * |u|^2)u$ in $\mathbb{R}^5$ with initial data in energy space H^1. If $M[u_0]E[u_0] < M[Q]E[Q]$, $\|u_0\|_2\|\nabla u_0\|_2 < \|Q\|_2\|\nabla Q\|_2$. Then the solution with initial data u_0 is global and scatters. Here Q is the ground state solution. Moreover, we show that if $M[u_0]E[u_0] < M[Q]E[Q]$, but $\|u_0\|_2\|\nabla u_0\|_2 > \|Q\|_2\|\nabla Q\|_2$, then the corresponding solution will blow up in finite time. The argument of this work is based on the linear profile decomposition, in the spirit of Kenig–Merle.

1. Introduction

Consider the Cauchy problem for the focusing Hartree equation

$$\mathrm{i}\partial_t u + \Delta u + F(u) = 0, \tag{1.1}$$

where $F(u) = (|\cdot|^{2-d} * |u|^2)u$, u is a complex valued function defined on some time–space slab $I \times \mathbb{R}^d$, $*$ denotes the convolution.

NLS with Hartree type nonlinearity $(|\cdot|^{2-d}*|u|^2)u$ describes the dynamics of the mean-field limits of many-body quantum systems, such as coherent states, condensates. In particular, it provides effective model for quantum systems with long-range interactions. In $d = 3$, the convolution kernel is $|x|^{-1}$, which represents Coulomb interactions, and which has infinite scattering length. When $d = 4$, the equation is L^2-critical. It is interesting from a mathematical point of view. In [1], Miao, Xu and Zhao proved that L^2-solution scatters when the initial data is strictly less than that of ground state. Li and Zhang [2] characterized the dynamics of minimal mass blow-up solutions. In this work, we make an investigation on scattering versus blow-up dichotomy in dimension 5.

There are also many other works devoted to NLS of Hartree type; see e.g., [3–9].

When $d = 5$, Eq. (1.1) is $\dot{H}^{1/2}$-critical, that is, both the equation and the $\dot{H}^{1/2}$-norm of initial data are preserved by the scaling

$$u_\lambda(t, x) = \lambda^{-2}u(\lambda^{-2}t, \lambda^{-1}x).$$

* Corresponding author.
E-mail addresses: gaoyanfang236@gmail.com (Y. Gao), wuhaigen@gmail.com (H. Wu).

0362-546X/$ – see front matter © 2010 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2010.04.033

We will consider in energy space H^1, in which the solution enjoys two conservation laws:

Energy conservation: $E[u(t)] := \frac{1}{2}\int_{\mathbb{R}^5} |\nabla u(t,x)|^2\,dx - \frac{1}{4}\iint_{\mathbb{R}^5\times\mathbb{R}^5} \frac{|u(x)|^2|u(y)|^2}{|x-y|^3}\,dxdy \equiv E[u_0]$;

Mass conservation: $M[u(t)] := \|u(t)\|_2^2 \equiv M[u_0]$.

By a function $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ is a *solution* to (1.1), it means that $u \in C_t^0 H_x^1(K\times\mathbb{R}^5) \cap L_t^3 L_x^{15/4}(K\times\mathbb{R}^5)$ for any compact $K \subset I$, and u obeys the Duhamel formula

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) + i\int_{t_0}^{t} e^{i(t-t')\Delta}F(u(t'))\,dt'$$

for all $t,\ t_0 \in I$. We call I the *life-span* of u. If I cannot be extended strictly larger, we say I is the *maximal life-span* of u, and u is a *maximal life-span solution*. If $I = \mathbb{R}$, then u is global.

Definition 1.1 (*Blow-up*). Let $u : I \times \mathbb{R}^5 \mapsto \mathbb{C}$ be a solution to (1.1). Say u blows up forward in time if there exists $t_1 \in I$ such that

$$\|u\|_{L_t^3 L_x^{15/4}([t_1,\sup I)\times\mathbb{R}^5)} = \infty;$$

and u blows up backward in time if there exists t_1 such that

$$\|u\|_{L_t^3 L_x^{15/4}((\inf I,t_1]\times\mathbb{R}^5)} = \infty.$$

For the Cauchy problem of (1.1), there is a stationary solution $e^{it}Q$ that is global but blows up both forward and backward. Here Q is the unique positive radial Schwarz solution to

$$-\Delta Q + Q = (|\cdot|^{-3} * |Q|^2)Q. \tag{1.2}$$

This Q is expected to be the threshold for scattering versus blow-up, which we will show in Section 4. Indeed, we shall show that the threshold is $M[Q]E[Q]$, which is invariant under our scaling. Authors in [10] first take this quantity to be a scattering threshold instead of $\|Q\|_{\dot{H}^{1/2}}$ for cubic NLS in dimension 3.

The recent progress in studying NLS or of Hartree type is due to a new and highly efficient approach based on a concentration–compactness idea to provide a linear profile decomposition. This approach arises from investigating the defect of compactness for the Strichartz estimates. Based on a refined Sobolev inequality, Kerrani [11] obtained a linear profile decomposition for solutions of free NLS with H_x^1 data. It was Kenig and Merle who first introduced Kerrani's linear profile decomposition to obtain scattering results. They treated the focusing energy-critical NLS in dimensions 3, 4, 5 in [12]. The defect of the compactness is due to the concentration phenomena. Using this decomposition, one can extract an almost periodic solution, of which mass/energy is concentrated at both spatial and frequency space. And thus make an reduction to the failure of scattering. For more excellent work using this technique, refer [13,14] and the references there in.

Adapted the ideas of Kerrani's, we can show a variant linear profile decomposition (Lemma 2.2) for H^1-subcritical problem. It has been proved by Duyckaers, Holmer and Roudenko in [10] for cubic NLS. We will give the proof in our context. From the compactness view, it means that the defect of compactness for Strichartz operators is only caused by the symmetry of time modulation and space translation.

Based on this linear profile decomposition, we prove our main result:

Theorem 1.1. *Let $u_0 \in H^1(\mathbb{R}^5)$ be radially symmetric, and let u be the corresponding solution to (1.1) with maximal life-span I. Assume $M[u]E[u] < M[Q]E[Q]$. If $\|u_0\|_2\|\nabla u_0\|_2 < \|Q\|_2\|\nabla Q\|_2$. Then u is global and scatters in H^1, i.e., there exists $\phi_+ \in H^1$ such that*

$$\lim_{t\nearrow\infty} \|u(t) - e^{it\Delta}\phi_+\|_{H^1} = 0.$$

If $\|u_0\|_2\|\nabla u_0\|_2 > \|Q\|_2\|\nabla Q\|_2$, assume in addition $xu_0 \in L^2$ or $u_0 \in H^1$ be radial. Then u blows up in finite time.

The finite time blow-up results of this kind for various equations appeared previously in such as [15]. The proof is based on the standard convexity method, calculating the second order derivative of $\|xu_0\|_2^2$ if the initial data has finite variance. When the initial data is just radial, we use a localized virial identity argument. It can be viewed as a smoothed variance.

The rest of the paper is organized as follows: In Section 2, we give some basic results and linear decomposition. In Section 3, we explore the properties of the ground state. In Section 4, we give a dichotomy of global existence versus blow-up. In Section 5, we proceed the concentration–compactness progress. In Section 6, we prove a rigidity theorem which leads to a contradiction with our concentration–compactness assumption.

2. Preliminaries

For $u \in H^1(\mathbb{R}^5)$, we will write

$$P(u) = \iint_{\mathbb{R}^5\times\mathbb{R}^5} \frac{|u(x)|^2|u(y)|^2}{|x-y|^3}\,dxdy.$$