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Topological properties of asymptotically stable sets

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ABSTRACT

Topological properties of the domain of attraction for dynamical systems are investigated. The main purpose of this paper is to prove that a compact, asymptotically stable attractor of a dynamical system defined on a locally compact metric space is a deformation retract of its domain of attraction, in a weak sense that is made precise. Under additional local assumptions, the attractor can be shown to be a retract, a deformation retract, or a strong deformation retract. The well known result that the domain of attraction of an asymptotically stable equilibrium is contractible follows as a corollary.

1. Introduction

In [1, Thm. 21], Eduardo Sontag has shown that the domain of attraction of a dynamical system with an asymptotically stable equilibrium point is contractible. In [2], it is proved that global asymptotic stability on a vector bundle implies contractibility of the base manifold. In particular, a continuous dynamical system on a state space that has the structure of a vector bundle on a compact manifold possesses no globally asymptotically stable equilibrium.

The main goal of this paper is to generalize Sontag's result [1, Thm. 21] for equilibria to compact, asymptotically stable sets. Our main result shows that a compact, asymptotically stable attractor of a continuous dynamical system defined on a locally compact metric space is a weak deformation retract of its domain of attraction in a sense that we define. A weak version of our result involving retract only has been stated in [3, Lemma 3.2]. We also show that, under additional topological assumptions, a weak deformation retract is a retract, a deformation retract, or a strong deformation retract. The additional assumptions are trivially satisfied in the case of an equilibrium, and hence [1, Thm. 21] follows as a corollary of our results.

We introduce the necessary notation and topological preliminaries in Section 2. Section 3 contains the main results. Finally, the conclusion is presented in Section 4.

2. Preliminaries

Throughout this paper, we consider a continuous, time-invariant semi-flow $\varphi : \mathbb{R}_{\geq 0} \times \mathcal{M} \rightarrow \mathcal{M}$ on a locally compact topological space $\mathcal{M}$. The semi-flow φ satisfies $\varphi(0, x) = x$ and $\varphi(t, \varphi(s, x)) = \varphi(t + s, x)$ for all $t, s \in \mathbb{R}_{\geq 0}$ and $x \in \mathcal{M}$. The topological space $\mathcal{M}$, called the state space, and the semi flow φ, also called the evolution function, together define a dynamical system, which we denote by $(\mathcal{M}, \varphi)$ (see for instance [4,3]).

For each $t \in \mathbb{R}_{\geq 0}$, we let φ_t denote the map $x \mapsto \varphi(t, x)$. Next, we recall the definition of stability.

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Definition 1. The set $\mathcal{K} \subset \mathcal{M}$ is *asymptotically stable* if the following two conditions hold.

1. $\mathcal{K}$ is *Lyapunov stable*, that is, for every open neighborhood $\mathcal{V} \subseteq \mathcal{M}$ of $\mathcal{K}$, there exists an open neighborhood $\mathcal{U} \subseteq \mathcal{M}$ of $\mathcal{K}$ such that $\varphi_t(\mathcal{U}) \subseteq \mathcal{V}$ for every $t \geq 0$.
2. $\mathcal{K}$ is *attractive*, that is, there exists an open neighborhood $\mathcal{W} \subseteq \mathcal{M}$ of $\mathcal{K}$ such that, for every $x \in \mathcal{W}$ and every open neighborhood $\mathcal{U} \subseteq \mathcal{M}$ of $\mathcal{K}$, there exists $T \geq 0$ such that $\varphi(t, x) \in \mathcal{U}$ for all $t > T$.

The *domain of attraction* of an asymptotically stable set $\mathcal{K} \subseteq \mathcal{M}$ is the set $\mathcal{A}$ of points x such that, for every open neighborhood $\mathcal{U}$ of $\mathcal{K}$, there exists $T > 0$ such that $\varphi(t, x) \in \mathcal{V}$ for all $t \geq T$.

It is well known that the domain of attraction $\mathcal{A}$ of a compact asymptotically stable set is open and *invariant*, that is, $\varphi_t(\mathcal{A}) = \mathcal{A}$ for all $t \geq 0$.

Next, we recall various different notions of retracts. A survey on the theory of retracts can be founded in [5].

Definition 2. Let $X \subseteq Y \subseteq \mathcal{M}$.

1. X is a *retract* of Y if there exists a continuous map $r : Y \to X$, called a *retraction*, such that $r(x) = x$ for all $x \in X$.
2. X is a *deformation retract* of Y if there exists a continuous *homotopy* $h : [0, 1] \times Y \to \mathcal{M}$ such that
$$h(0, y) = y,$$
$$h(1, y) \in X,$$
$$h(1, x) = x,$$
for all $y \in Y$ and $x \in X$.
3. X is a *strong deformation retract* of Y if there exists a continuous *homotopy* $h : [0, 1] \times Y \to \mathcal{M}$ such that
$$h(0, y) = y,$$
$$h(1, y) \in X,$$
$$h(t, x) = x,$$
for all $y \in Y$, $x \in X$ and $t \in [0, 1]$.
4. X is a *weak deformation retract* of Y if every open neighborhood $U \subseteq \mathcal{M}$ of X contains a strong deformation retract V of Y such that $X \subseteq V$.

A set $Y \subseteq \mathcal{M}$ is *contractible* if there exists $x \in \mathcal{M}$ such that the singleton set $X = \{x\}$ is a strong deformation retract of Y. We will also need the following definition.

Definition 3. $X \subseteq \mathcal{M}$ is a neighborhood retract (respectively, neighborhood deformation retract, strong neighborhood deformation retract) of $\mathcal{M}$ if there exists an open neighborhood $Y \subseteq \mathcal{M}$ of X such that X is a retract (respectively, deformation retract, strong deformation retract) of Y.

Our first result relates Definitions 2 and 3.

Theorem 4. *Suppose $X \subseteq \mathcal{M}$ is a weak deformation retract of $Y \subseteq \mathcal{M}$. Then the following statements hold.*

1. *If X is a neighborhood retract of $\mathcal{M}$, then X is a retract of Y.*
2. *If X is a neighborhood deformation retract of $\mathcal{M}$, then X is a deformation retract of Y.*
3. *If X is a strong neighborhood deformation retract of $\mathcal{M}$, then X is a strong deformation retract of Y.*

Proof. First, note that since X is a weak deformation retract of Y, for every neighborhood $U \subseteq \mathcal{M}$, there exist a set $V \subseteq U$ containing X, and a continuous map $h : [0, 1] \times Y \to V$ such that $h(0, y) = y$, $h(1, y) \in V$ and $h(t, v) = v$ for all $y \in Y$, $v \in V$ and $t \in [0, 1]$.

To prove (1), suppose X is a neighborhood retract of $\mathcal{M}$. Then there exists an open neighborhood $U \subseteq \mathcal{M}$ of X and a continuous map $s : U \to X$ such that $s(x) = x$ for all $x \in X$. Upon choosing the set V and the homotopy h as above, the map $r : Y \to X$ given by $r(y) = s(h(1, y))$ can easily be verified to be a retraction. Hence (1) follows.

Next,to prove (2), suppose X is a neighborhood deformation retract of $\mathcal{M}$. Then there exists an open neighborhood $U \subseteq \mathcal{M}$ of X and a continuous deformation retraction $s : [0, 1] \times U \to X$ such that $s(0, u) = u$, $s(1, u) \in X$, and $s(1, x) = x$ for all $u \in U$ and $x \in X$. On choosing the set V and the homotopy h as in the first paragraph, it follows that the map $r : [0, 1] \times Y \to X$ given by
$$\begin{aligned} r(t, y) &= h(2t, y), \quad t \in \left[0, \frac{1}{2}\right], y \in Y, \\ &= s\left(2t - 1, h(1, y)\right), \quad t \in \left(\frac{1}{2}, 1\right], y \in Y, \end{aligned}$$
is continuous and satisfies $r(0, y) = y$, $r(1, y) \in X$ and $r(1, x) = x$ for all $y \in Y$ and $x \in X$. Hence X is a deformation retract of Y.

The proof of (3) follows from the proof of (2) above by noting that, in the case where X is a strong neighborhood deformation retract of $\mathcal{M}$, the homotopy s may be chosen to additionally satisfy $s(t, x) = x$ for all $x \in X$. Consequently, the homotopy r constructed above will also satisfy $r(t, x) = x$ for all $x \in X$. Hence X is a strong deformation retract of Y. □