Available online at www.sciencedirect.com
ScienceDirect
Nonlinear Analysis 69 (2008) 4252–4265
Nonlinear Analysis
www.elsevier.com/locate/na

Review

Stability of periodic nonlinear Schrödinger equation

J.P. Borgna*

Dpto. Mat., FCEyN-Univ. de Buenos Aires, Ciudad Univ. Pb. I, Argentina

Received 10 January 2007; accepted 9 November 2007

Abstract

In this manuscript, we study the existence of steady states of the periodic nonlinear Schrödinger equation in dimension one and we prove the stability of the solutions with initial data close to the ground state profile, when the potential parameter σ is small enough.

Keywords: Ground states existence; Orbital stability; Perturbation method

Contents

0. Introduction

This work is devoted to the study of the initial value problems associated to the periodic nonlinear Schrödinger equation (PNLS)

$$\begin{cases} \partial_t \phi = \mathrm{i}\partial_x^2 \phi + \mathrm{i}\,|\phi|^{2\sigma}\,\phi & (x,t) \in (0,1) \times \mathbb{R}_{>0} \\ \phi(0,t) = \phi(1,t) = 0 & t > 0 \\ \partial_x \phi(0,t) = \partial_x \phi(1,t) & t > 0 \\ \phi(x,0) = \phi_0(x) & 0 \le x \le 1 \end{cases} \tag{PNLS}$$

with $\sigma > 0$ and ϕ a function in a suitable Sobolev space. Problem (PNLS) arises in the propagation of electromagnetic waves in a nonlinear medium, as a laser beam in an optical fiber. In the physics literature (for instance see [1]) the

* Tel.: +54 11 4576 3335; fax: +54 11 4576 3335.
E-mail address: jpborg@dm.uba.ar.

0362-546X/$ - see front matter
doi:10.1016/j.na.2007.11.016

presence of potential barriers that confine the charged particles to a bounded domain often seems associated with the periodic problem and is also known that the wave function is almost zero in the region of the barrier, so it is natural to ask that the solution of the equation is equal to zero on the border, as well as asserting the periodicity conditions.

Well-posedness and existence of solutions for the initial value problem associated to (PNLS) were studied by Bourgain [3] and Kavian [9]. Bourgain's work gives local existence of solution in time and Kavian provided global existence using conservation laws. In addition Kavian proved, for $\sigma \geq 2$, that there exist blowing-up solutions.

The problem with power nonlinearity with $\mathbb{R}^n$ domain (problem NLS) was studied by many authors. Ginibre and Velo proved that NLS has a unique global solution of class $C\left([0,+\infty), H^1\right)$ if the initial data $\phi_0(x) \in H^1(\mathbb{R}^n)$ (see [6]). Also Kato proved the local existence and uniqueness for a general nonlinear potential F (see [8]). For a summary of results we refer the reader to [4,5].

Ground state existence for the problem in $\mathbb{R}^n$ was studied by Strauss in [10], and the orbital stability in this case has been study in [11], where the author used a Gagliardo–Nirenberg functional and rescaling as main tools. In the periodic problem it is not possible to apply rescaling, then the orbital stability has been proved, in an appropriate soliton neighborhood, using perturbation theory.

This article is structured as follows. In Section 1 we present the existence of ground states, by means of potential methods, and some ground state properties. In Section 2 we present the concepts of orbit, distance and the Lyapunov function, and it is seen that the stability relies on a suitable lower bound on the second variation of the Lyapunov function. This lower bound has two parts. In Section 3 we used the analysis of a constrained variational problem carried out in [11] for obtaining the bound for the first part. In Section 4 we use the perturbation theory by the second part, and it consists of the main part of this work. In Section 5 we outline a proof of the orbital stability.

1. Ground state

We will denote by $\mathcal{W} \in L^2[0,1]$ the functions $\phi(x)$ such that $\phi(1-x) = -\phi(x)$ for all $x \in [0,1]$ (ϕ is odd with respect to the midpoint $x = 1/2$).

Remark 1. If the initial data $\phi_0(x) \in \mathcal{W}$, the flux of Eq. (PNLS) holds in $\mathcal{W}$.

Proof. We suppose that $\phi(x,t)$ and $\psi(x,t)$ are solutions of (PNLS) with initial data $\phi_0(x)$ and $\psi_0(x) = -\phi_0(1-x)$ respectively. If $\phi_0(x) \in \mathcal{W}$, using (PNLS) and the uniqueness of solution, we conclude that $\phi(x,t) = \psi(x,t)$. □

In this section we will prove that there is a profile of the ground state solution of (PNLS) in $\mathcal{W}$. Thus, how we will consider solutions of the Eq. (PNLS) as perturbations of the ground states, we can look for them in $\mathcal{W}$.

A ground state of the (PNLS) is a solution of the stationary problem

$$\begin{cases} R'' - ER + R^{2\sigma+1} = 0 & \text{if } x \in (0,1) \\ R(0) = R(1) = 0 \\ R'(0) = R'(1). \end{cases} \tag{1.1}$$

In this section we prove the existence of the ground state or soliton solution and its other properties. We use potential theory instead of the fact that the soliton R is a minimum of the Gagliardo–Nirenberg functional as in [10,11], it is due to the fact that the profile of the minimum of this functional could need a rescaling because it can concentrate, but this is not possible if we have a bounded domain.

Proposition 1. *For all $E > 0$ and $\sigma > 0$ there exists a nontrivial solution $R_\sigma(x)$ of (1.1) satisfying the following properties*

(1) *R_σ is a periodic function, with period length one, and $R_\sigma(0) = R_\sigma(1/2) = R_\sigma(1) = 0$.*
(2) *$R_\sigma \in \mathcal{W}$, is positive in $(0, 1/2)$, is increasing in $[0, 1/4]$ and symmetric in $[0, 1/2]$ respect to the midpoint and it has a maximum at this point.*
(3) *$R_\sigma \in C^\infty \cap H^1_0[0, 1/2]$.*

Proof. Eq. (1.1) takes the form

$$\left(R'\right)^2 - ER^2 + \frac{1}{\sigma+1} R^{2\sigma+2} = C \tag{1.2}$$