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On global solution of an initial boundary value problem for a class of damped nonlinear equations

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Abstract

In this paper, we consider the following problem

$$u_{tt} - \alpha \triangle u_t + \triangle^2 u - \triangle u = f(u), \quad x \in \Omega, t > 0,$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega,$$
$$\triangle u(x,t)|_{\partial\Omega} = u(x,t)|_{\partial\Omega} = 0, \quad t \geq 0,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Under some assumptions of the initial data $u_0(x)$, $u_1(x)$ and the nonlinear function $f(u)$, the existence of global weak solutions and global strong solutions are obtained by means of the potential well method.

Keywords: Initial boundary value problem; Potential wells; Global weak solution; Global strong solution

1. Introduction

In 1968, Sattinger [14] introduced the potential well method to show the existence of global solutions for the nonlinear hyperbolic equations that do not possess positive definite energy. Since then, many authors [4,8,13,15,16] have applied the potential well method to investigate the existence and nonexistence of global solutions to the initial boundary value problems for various nonlinear evolution equations.

In [17], Webb introduced and investigated the following second-order nonlinear initial boundary value problem

$$u_{tt} - \alpha \triangle u_t - \triangle u = f(u) \tag{1.1}$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{1.2}$$
$$\triangle u(x,t)|_{\partial\Omega} = u(x,t)|_{\partial\Omega} = 0, \quad t \geq 0, \tag{1.3}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. In [9–11,17] Eq. (1.1) was studied under different assumptions of $f(u)$.

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In [3], the following fourth-order evolution equation is analyzed

$$u_{tt} + \triangle^2 u - \sum_{i=1}^{n} (\sigma_i(u_{x_i}))_{x_i} = f(u) \tag{1.4}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. The authors defined and used a potential well and several invariant and positive invariant sets to study the characteristics of blowup, boundedness and convergence as $t \to \pm\infty$. In [12], an initial boundary value problem for fourth-order wave equations with nonlinear strain and source terms is considered. By introducing a family of potential wells and proving the invariance of some sets, the authors obtained a threshold result of global existence and nonexistence. Various studies on the solution and properties of fourth-order wave equations have also been conducted and can be found in references [1–3,5–7,12,18,19].

Motivated by the previous work, in this paper, we are concerned with the following fourth-order nonlinear evolution equation, with a damped term, subject to initial conditions (1.2) and boundary conditions (1.3)

$$u_{tt} - \alpha \triangle u_t + \triangle^2 u - \triangle u = f(u), \quad x \in \Omega, t > 0, \tag{1.5}$$

where $\alpha > 0$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, and $\triangle u_t$ is a damped term, $f(u)$ is a given nonlinear function, $u_0(x)$ and $u_1(x)$ are given initial functions.

In comparison with Eq. (1.1), the equation to be investigated includes an additional term describing the fourth-order effects. The nature of the equation to be investigated is also different from the fourth order Eq. (1.4) due to the addition of the damping term. Hence, the previous results do not cover the problem in question. In this paper, we apply the potential well method to establish the conditions under which the initial boundary value problem in question has global weak solutions and global strong solutions. The rest of the paper is organized as follows. In section two, we define the potential well for the initial boundary value problem in question and investigate its properties. Based on the results obtained in section two, the existence of global weak solutions and global strong solutions are established respectively in sections three and four.

We should also address here that throughout this paper, we denote $\|\cdot\|_{L^p(\Omega)}$ by $\|\cdot\|_p$, $\|\cdot\|_2 = \|\cdot\|$, and $(u, v) = \int_\Omega uv\mathrm{d}x$.

2. The potential well and its properties

We assume that $f(u) \in C$, $f(u)u \geq 0$ and

$$|f(u)| < a|u|^p, \tag{2.1}$$

where $1 < p < \infty$ if $n = 1, 2$ and $1 < p \leq \frac{n+2}{n-2}$ if $n \geq 3$.

For the initial boundary value problem consisting of (1.5) and (1.2)–(1.3), we define

$$J(u) = \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\triangle u\|^2 - \frac{a}{p+1}\|u\|_{p+1}^{p+1}, \tag{2.2}$$

$$I(u) = \|\nabla u\|^2 + \|\triangle u\|^2 - a\|u\|_{p+1}^{p+1}, \tag{2.3}$$

and

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\triangle u\|^2 - \int_\Omega F(u)\mathrm{d}x, \tag{2.4}$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$.

We also define the potential well

$$W = \{u \in H^2(\Omega) \cap H_0^1(\Omega) | I(u) > 0, J(u) < d\} \cup \{0\} \tag{2.5}$$

and a set outside the potential well

$$V = \{u \in H^2(\Omega) \cap H_0^1(\Omega) | I(u) < 0, J(u) < d\}, \tag{2.6}$$