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# Strong convergence theorems for a finite family of asymptotically nonexpansive mappings and semigroups

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**Abstract**

Strong convergence theorems are obtained for a finite family of asymptotically nonexpansive mappings and semigroups by the modified Mann method.


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## 1. Introduction

Let $K$ be a nonempty closed convex subset of a Hilbert space $H$. A mapping $T : K \to K$ is said to be *nonexpansive* if for all $x, y \in K$ we have $\|Tx - Ty\| \leq \|x - y\|$. It is said to be *asymptotically nonexpansive* [2] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all integers $n \geq 1$ and all $x, y \in K$. The set of fixed points of $T$ is denoted by $F(T)$.

One parameter family $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\}$, where $\mathbb{R}^+$ denotes the set of nonnegative real numbers, is said to be *a (continuous) Lipschitzian semigroup on* $K$ [16] *of mappings* from $K$ into $K$ if the following conditions are satisfied:

(1) $T(0)x = x$ for all $x \in K$;
(2) $T(s + t) = T(s)T(t)$ for all $s, t \in \mathbb{R}^+$;
(3) for each $t > 0$, there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that $\|T(t)x - T(t)y\| \leq L_t \|x - y\|$, $x, y \in K$;
(4) for each $x \in K$, the mapping $T(.)x$ from $\mathbb{R}^+$ into $K$ is continuous.

A Lipschitzian semigroup $\mathcal{T}$ is called *nonexpansive (or contractive)* if $L_t = 1$ for all $t > 0$, and *asymptotically nonexpansive* if $\limsup_{t\to\infty} L_t \leq 1$, respectively. Let $F(\mathcal{T})$ denote the common fixed point set of the semigroup $\mathcal{T}$, i.e., $F(\mathcal{T}) := \{x \in K : T(t)x = x, \forall t > 0\}$. Notice that for an asymptotically nonexpansive semigroup $\mathcal{T}$, we can

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always assume that the Lipschitzian constants $\{L_t\}_{t>0}$ are such that $L_t \geq 1$ for each $t > 0$, $L_t$ is non-increasing in $t$, and $\lim_{t\to\infty} L_t = 1$; otherwise we replace $L_t$, for each $t > 0$, with $\bar{L_t} := \max\{\sup_{s\geq t} L_s, 1\}$.

Construction of fixed points of nonexpansive mappings and asymptotically nonexpansive mappings (and of common fixed points of nonexpansive semigroups and asymptotically nonexpansive semigroups) is an important subject in nonlinear operator theory and its applications, in particular, in image recovery and signal processing (see, e.g., [1,9,17]). Fixed point iteration process for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann [6] and Ishikawa [3] iteration process have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see e.g., [3,6,10–13,15]. However, Mann and Ishikawa iteration processes have only weak convergence even in Hilbert spaces, (see, e.g., [6,3]).

Some attempts to modify the Mann iteration method so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [8] proposed the following modification of the Mann iteration method for a nonexpansive mapping $T$ in a Hilbert space $H$:

$$\begin{cases} x_0 \in K \text{ chosen arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{v \in K : \|y_n - v\| \leq \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \end{cases} \tag{1.1}$$

where $P_K$ denotes the metric projection from $H$ onto a closed convex subset $K$ of $H$. They proved that if the sequence $\{\alpha_n\}$ bounded above from one, then $\{x_n\}$ defined by (1.1) converges strongly to $P_{F(T)}$. Moreover, they introduced and studied an iteration process of a nonexpansive semigroup $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\}$ in a Hilbert space $H$:

$$\begin{cases} x_0 \in K \text{ chosen arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\left(\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(u)x_n \mathrm{d}u\right), \\ C_n = \{v \in C : \|y_n - v\| \leq \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0). \end{cases} \tag{1.2}$$

Recently, Kim and Xu [4] adapted the iteration (1.1) to asymptotically nonexpansive mappings in a Hilbert space $H$:

$$\begin{cases} x_0 \in K \text{ chosen arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \end{cases} \tag{1.3}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(\mathrm{diam}(K))^2 \to 0$ as $n \to \infty$. They also proved that if $\alpha_n \leq a$ for all $n$ and for some $0 < a < 1$, then the sequence $\{x_n\}$ generated by (1.3) converges strongly to $P_{F(T)}(x_0)$. Moreover, they modified an iterative method (1.2) to the case of an asymptotically nonexpansive semigroup $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\}$ in a Hilbert space $H$:

$$\begin{cases} x_0 \in K \text{ chosen arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\left(\dfrac{1}{t_n}\displaystyle\int_0^{t_n} T(u)x_n \mathrm{d}u\right), \\ C_n = \{v \in C : \|y_n - v\|^2 \leq \|x_n - v\|^2 + \theta_n\}, \\ Q_n = \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}(x_0), \end{cases} \tag{1.4}$$

where $\theta_n = (1-\alpha_n)[\frac{1}{t_n}(\int_0^{t_n} L_u \mathrm{d}u)^2 - 1](\mathrm{diam}(K))^2 \to 0$, as $n \to \infty$. Under the same conditions of the sequence $\{\alpha_n\}$ and $\{t_n\}$, using boundedness of the nonempty closed convex subset $K$ and the Lipschitzian constant $L_t$ of the mapping $T(t)$, where $L_t$ is bounded and measurable, they proved that the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{F(\mathcal{T})}(x_0)$.