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Property of positive solutions and its applications for Sturm–Liouville singular boundary value problems☆

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Abstract

This paper is concerned with the property of the positive solutions for Sturm–Liouville singular boundary value problems with the linear conditions. We obtain a relation between the solutions and Green's function. It implies a necessary condition for the $C^1[0, 1]$ positive solutions. We apply the result to conclude that the given equation has no $C^1[0, 1]$ positive solutions.

MSC: 34B15; 34B25

Keywords: Singular Sturm–Liouville problem; Property of solution

1. Instruction and main results

There are many studies on two-point boundary value problems of singular second-order nonlinear ordinary differential equations. They use the method of upper and lower solutions to gain the existence of continuous solutions (see [1]), or they change the equation into Fredholm integral equation and obtain $C^1[0, 1]$ positive solutions with the fixed-points theory (the kernel of integral equation is Green's function of corresponding differential equation) (see [2–7]). In addition, it is important about the property of solutions. In this paper, we study the property of solutions to Sturm–Liouville nonlinear singular boundary value problems and imply the relation between the solutions and its Green's function. At last, we apply the result to conclude that the given equation has no $C^1[0, 1]$ positive solutions. The restriction on nonlinear term f we study is very mild. So the case where the theory could be used is greater.

We study the properties of positive solutions for singular Sturm–Liouville problem

$$\begin{cases} Lu = f(t, u), \quad 0 < t < 1 \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{1}$$

where $Lu = -(p(t)u')' + s(t)u$, $p(t) \in C^1([0, 1], (0, \infty))$, $s(t) \in C([0, 1], [0, \infty))$, $\alpha, \beta, \delta, \gamma$ are nonnegative real numbers and $d := \alpha\gamma + \alpha\delta + \beta\gamma > 0$.

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$f(t,u)$: $J \times R^+ \to R^+$, it may be singular at $t = 0$, $t = 1$ or (and) $u = 0$.

Let $J = (0, 1)$, $I = [0, 1]$, $R^+ = (0, +\infty)$. In the following discussions, we will say that w is a solution of (1) if a function $w(t) \in C(I) \cap C^2(J)$ satisfies the problem (1). The solution of (1) is also called a $C(I)$ solution. In addition, if there is a solution $\omega(t) \in C^1(I)$, i.e. both $w'(0+)$ and $w'(1-0)$ exist, we call it a $C^1(I)$ solution. A solution $w(t)$ of (1) is said to be positive if $w(t) > 0$ for $t \in J$.

We give the following postulations:

$$\text{(H)} \qquad \liminf_{(t,u)\to(0^+,0^+)} f(t,u) > 0 \quad (\text{or } +\infty), \qquad \liminf_{(t,u)\to(1^-,0^+)} f(t,u) > 0 \quad (\text{or } +\infty).$$

The main results of this paper are as follows.

Theorem 1.1. *Suppose that $s(t) \equiv 0$ or $s(t) \not\equiv 0$ and $f(t,u)$ satisfy (H), $w(t)$ is a $C^1(I)$ positive solution of the boundary problem* (1)*. Then there exist real numbers $0 < m < 1 < M$ such that*

$$me(t) \le w(t) \le Me(t), \quad \forall\, t \in I, \tag{2}$$

*where $e(t) = G(t,t)$, $G(t,s)$ is **Green's** function of the problem* (1) *(when $f(t,u) = 0$).*

Theorem 1.2. *Suppose that $w(t)$ is a $C(I)$ positive solution of the boundary problem* (1)*. Then there exist real numbers $0 < m < 1$ such that*

$$me(t) \le w(t), \quad \forall\, t \in I.$$

Remark 1. Notice that we do not assume $f(t,u)$ to be continuous, or bounded or integrable in Theorems 1.1 and 1.2.

We know that [8], **Green**'s function of the boundary value problem (1) is given by

$$G(t,s) = \begin{cases} \dfrac{1}{\Delta} B(t)D(s), & 0 \le t \le s \le 1 \\ \dfrac{1}{\Delta} B(s)D(t), & 0 \le s < t \le 1, \end{cases} \tag{3}$$

where $B(t) \in C^2(I)$, $D(t) \in C^2(I)$ satisfy

$$LB = 0, \quad B(0) = \beta, \quad B'(0) = \alpha \tag{4}$$

$$LD = 0, \quad D(1) = \delta, \quad D'(1) = -\gamma \tag{5}$$

respectively. Δ is a constant which satisfies

$$\frac{p(t)(B(t)D'(t) - D(t)B'(t))}{\Delta} = -1. \tag{6}$$

Concerning the functions in (3)–(6), we have the following lemma.

Lemma 1.1. *Suppose that $B(t)$, $D(t)$ and $G(t,s)$, Δ as* (3)–(6)*, then we have*

(i) *$B(t)$ is monotone increasing on I, $B(t) > 0, t \in (0, 1]$;*
(ii) *$D(t)$ is monotone decreasing on I, $D(t) > 0, t \in [0, 1)$;*
(iii) $\Delta > 0$.

Obviously

$$e(t) = \frac{1}{\Delta} B(t)D(t), \quad \forall\, t \in I. \tag{7}$$

For the Proof of the conclusions (i)–(iii) see [9].