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Radial minimizers of p-Ginzburg–Landau type with $p \in (n-1, n)$

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Abstract

The author studies the asymptotic behavior of the radial minimizer of the p-Ginzburg–Landau functional with non-vanishing Dirichlet boundary condition in the case of $p \in (n-1, n)$, where n is the dimension. The location of the zeros of the radial minimizer is discussed. Moreover the $C^{1,\alpha}$ convergence of the radial minimizer is proved.

MSC: 35J70; 49K20

Keywords: p-Ginzburg–Landau functional; Radial minimizer; Asymptotic behavior; Location of zeros; Uniqueness; Convergence rate

1. Introduction

In this paper, we are concerned with the asymptotic behavior of the minimizer u_ε of the p-Ginzburg–Landau type functional

$$E_\varepsilon(u, B) = \frac{1}{p}\int_B |\nabla u|^p \mathrm{d}x + \frac{1}{4\varepsilon^p}\int_B (1-|u|^2)^2 \mathrm{d}x$$

in the function class $W = \{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B, \mathbf{R}^n); f(1) = M, r = |x|\}$, where the dimension $n \geq 2$ and $B = \{x \in \mathbf{R}^n; |x| < 1\}$. By using the direct method in the calculus of variations, we can see that the minimizer u_ε exists, and we call it the *radial minimizer*.

When $p = n = 2$, many papers studied this problem (cf. [3,8,12]). It turns out that the zero of u_ε is the origin 0; and as $\varepsilon \to 0$, there holds for some $\alpha \in (0, 1)$,

$$u_\varepsilon(x) \to \frac{x}{|x|}, \quad \text{in } C^{1,\alpha}_{\mathrm{loc}}(\overline{B} \setminus \{0\}). \tag{1.1}$$

In addition, the uniqueness and the multiplicity of the minimizer u_ε were discussed. When $p = 2 < n$, in [1,5], the authors also studied the properties of the radial minimizer.

When $p = n \geq 2$, it was proved in [9] that as $\varepsilon \to 0$,

$$u_\varepsilon(x) \rightharpoonup \frac{x}{|x|}, \quad \text{weakly in } W^{1,p}_{\mathrm{loc}}(\overline{B} \setminus \{0\}). \tag{1.2}$$

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When $p < n$, the convergence (1.2) for the general minimizer was obtained in [14]. All the results above were derived under the assumption of $M = 1$. If $M \neq 1$, we recall the results in [2,11]. The singularity of the energy functional appears not only at the origin, but also near the boundary.

We always assume that $p \in (n-1, n)$ and $M \in (0, 1)$ in this paper. As in [3,9,11], we also consider the asymptotic properties of the radial minimizer when ε goes to zero. The motivation of researching the behavior comes mainly from the pure mathematical point of view, namely, the study of p-harmonic map (cf. [3,9]). We will research p-harmonic maps by investigating the p-energy functional with a penalization. The p-Ginzburg–Landau is an appropriate model. To consider a simple p-harmonic map $\frac{x}{|x|}$, (in fact, it is also a p-energy minimizer (cf. [7,10])), we will establish the relation between the radial minimizer and the map $\frac{x}{|x|}$, to understand the properties of p-harmonic map.

If we denote $V = \{f \in W^{1,p}_{\text{loc}}(0, 1]; r^{\frac{n-1}{p}} f_r, r^{(n-1-p)/p} f \in L^p(0, 1), f(1) = M\}$, then $V = \{f(r); u(x) = f(r)\frac{x}{|x|} \in W\}$ and it is a subset of $C[0, 1]$. Substituting $u(x) = f(r)\frac{x}{|x|}$ into $E_\varepsilon(u, B)$, we obtain $E_\varepsilon(u, B) = |S^{n-1}|E_\varepsilon(f, [0, 1])$, where

$$E_\varepsilon(f, [a, b]) = \int_a^b \left[\frac{1}{p}(f_r^2 + (n-1)r^{-2}f^2)^{p/2} + \frac{1}{4\varepsilon^p}(1 - f^2)^2\right] r^{n-1}\mathrm{d}r.$$

This shows that $u = f(r)\frac{x}{|x|} \in W$ is the minimizer of $E_\varepsilon(u, B)$ if and only if $f(r) \in V$ is the minimizer of $E_\varepsilon(f, [0, 1])$. Applying the direct method in the calculus of variations, we can see that the functional $E_\varepsilon(u, B)$ achieves its minimum by a function $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$, hence $f_\varepsilon(r)$ is the minimizer of $E_\varepsilon(f, [0, 1])$.

We will prove the result as (1.2) in Section 2.

Theorem 1.1. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Then*

$$\lim_{\varepsilon\to 0} u_\varepsilon = \frac{x}{|x|}, \quad \text{in } W^{1,p}_{\text{loc}}(B, \mathbf{R}^n).$$

In the case of $p \in (n-1, n)$ and $M = 1$, the conclusion can be seen easily by the uniform estimate

$$E_\varepsilon(u_\varepsilon, B) \le E_\varepsilon\left(\frac{x}{|x|}, B\right) = \frac{|S^{n-1}|}{p(n-p)} \tag{1.3}$$

and the argument of the weak low semi-continuity of $\int_B |\nabla u|^p \mathrm{d}x$. However, the proof of Theorem 1.1 is not trivial since (1.3) does not hold when $M \neq 1$. In order to obtain the estimate of $\|\nabla u_\varepsilon\|_{L^p}$, firstly, by a simple construction it is not difficult to see that $E_\varepsilon(u_\varepsilon, B) \le C(\varepsilon^{1-p} + 1)$ with the absolute constant $C > 0$. Next, by improving the exponent of ε from $1 - p$ to $n - p$ step by step (the domain B shrinks to the compact subset K of $B \setminus \{0\}$ at the same time), we may deduce $E_\varepsilon(u_\varepsilon, K) \le C$. Finally, we may obtain the conclusion by the argument of the weak low semi-continuity.

We will discuss the location of zeros and the uniqueness of the radial minimizer in Section 3. The following result is based on the idea of distinguishing a class of *bad balls* from all the balls covering B (cf. [3]).

Theorem 1.2. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Then as $\varepsilon \to 0$, the zeros of $u_\varepsilon(x)$ are located near the origin 0 and the boundary ∂B. In addition, u_ε is unique for any given $\varepsilon \in (0, \varepsilon_0)$ as long as ε_0 is sufficiently small.*

Next, we will estimate the convergence rate of the module f_ε to 1.

Theorem 1.3. *Assume that $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{r}$ is a radial minimizer of $E_\varepsilon(u, B)$. If $p \in (p_*, n)$, where $p_* = n - 1$ when $n \ge 3$; $p_* = \frac{\sqrt{17}-1}{2}$ when $n = 2$. Then for any $T > 0$, there is $C > 0$ which is independent of $\varepsilon \in (0, \varepsilon_0)$, such that*

$$\|1 - f_\varepsilon\|_{W^{1,p}(T,1-T)} \le C\varepsilon, \quad \text{as } n \ge 3; \tag{1.4}$$

$$\|1 - f_\varepsilon\|_{C(T,1-T)} \le C\varepsilon^p, \quad \text{as } n \ge 2. \tag{1.5}$$

At last, in Section 5 we will prove the $C^{1,\alpha}$ convergence of u_ε.