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Causal functional differential equations in Banach spaces

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Abstract

The aim of this paper is to establish the existence of solutions and some properties of set solutions for a Cauchy problem with causal operator in a separable Banach space.

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1. Introduction and preliminaries

The study of functional equations with causal operators has seen a rapid development in the last few years and some results are assembled in a recent monograph [2]. The term causal operators is adopted from engineering literature and the theory of these operators has the powerful quality of unifying ordinary differential equations, integro-differential equations, differential equations with finite or infinite delay, Volterra integral equations, and neutral functional equations, to name but a few.

Let E be a real separable Banach space with norm $\|\cdot\|$. For $x \in E$ and $r > 0$ let $B_r(x) := \{y \in E; \|y - x\| < r\}$ be the open ball centered at x with radius r, and let $B_r[x]$ be its closure. If $\sigma > 0$, we denote by $\mathcal{C}([-\sigma, b), E)$ the Banach space of continuous bounded functions from $[-\sigma, b)$ into E and we denote by $\mathcal{C}_\sigma$ the space $\mathcal{C}([-\sigma, 0], E)$ with the norm $\|\varphi\|_\sigma = \sup_{-\sigma \le s \le 0} \|\varphi(s)\|$. By $L^p_{\mathrm{loc}}([0, b), E)$, $1 \le p \le \infty$, we denote the space of all functions which are L^p-Bochner integrable on each compact interval of $[0, b)$.

By $\alpha(A)$, we denote the Hausdorff measure of non-compactness of nonempty bounded set $A \subset E$, defined as follows [1,9]:

$$\alpha(A) = \inf\{\varepsilon > 0; A \text{ admits a finite cover by balls of radius } \le \varepsilon\}.$$

This is equivalent to the measure of non-compactness introduced by Kuratowski (see [1]).

If $\dim(A) = \sup\{\|x - y\|; x, y \in A\}$ is the diameter of the bounded set A, then we have that $\alpha(A) \le \dim(A)$ and $\alpha(A) \le 2d$ if $\sup_{x \in A} \|x\| \le d$. We recall some properties for α (see [9]).

If A, B are bounded subsets of E and $\overline{A}$ denotes the closure of A, then

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(1) $\alpha(A) = 0$ if and only if $\overline{A}$ is compact;
(2) $\alpha(A) = \alpha(\overline{A}) = \alpha(\overline{co}(A))$;
(3) $\alpha(\lambda A) = |\lambda|\alpha(A)$ for every $\lambda \in \mathbb{R}$;
(4) $\alpha(A) \leq \alpha(B)$ if $A \subset B$;
(5) $\alpha(A + B) = \alpha(A) + \alpha(B)$.

Also, we recall the following lemma due to Mönch ([10], Proposition 1.6).

Lemma 1.1. *Let $\{u_m(\cdot)\}_{m\geq 1}$ be a bounded sequence of continuous functions from $[0, T]$ into E. Then, $\beta(t) = \alpha(\{u_m(t); m \geq 1\})$ is measurable and*

$$\alpha\left(\left\{\int_0^T u_m(t)\mathrm{d}t; m \geq 1\right\}\right) \leq \int_0^T \beta(t)\mathrm{d}t. \quad \square$$

We recall that, by a Kamke function we mean a function $g : [0, b) \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the Carathéodory condition ($g(\cdot, w)$ is a measurable function for each fixed $w \in \mathbb{R}_+$ and $g(t, \cdot)$ is continuous for each fixed $t \in [0, b)$), $g(t, 0) = 0$ for a.e. $t \in [0, b)$, and such that $w(t) \equiv 0$ is only one solution of

$$w(t) \leq \int_0^t g(s, w(s))\mathrm{d}s$$

with $w(0) = 0$.

Definition 1.1. Let $\sigma \geq 0$. An operator $Q : \mathcal{C}([-\sigma, b), E) \to L^p_{\text{loc}}([0, b), E)$ is a causal operator if, for each $\tau \in [0, b)$ and for all $u(\cdot), v(\cdot) \in \mathcal{C}([-\sigma, b), E)$, with $u(t) = v(t)$ for every $t \in [-\sigma, \tau]$, we have $(Qu)(t) = (Qv)(t)$ for a.e. $t \in [0, \tau]$.

Two significant examples of causal operators are: the Niemytzki operator

$$(Qu)(t) = f(t, u(t))$$

and the Volterra integral operator

$$(Qu)(t) = g(t) + \int_0^t k(t, s) f(s, u(s))\mathrm{d}s.$$

For other concrete examples which serve to illustrate that the class of causal operators is very large, we refer the reader to the monograph [2].

In this paper, we consider the following Cauchy problem:

$$\begin{cases} u'(t) = (Qu)(t) & \text{a.e. } t \in [0, b) \\ u|_{[-\sigma,0]} = \varphi \in \mathcal{C}_\sigma, \end{cases} \tag{1.1}$$

where Q is a causal operator that satisfies the following conditions:

(h_1) Q is continuous;
(h_2) for each $r > 0$, there exists $\mu(\cdot) \in L^p_{\text{loc}}([0, b), \mathbb{R}_+)$ such that, for each $u(\cdot) \in \mathcal{C}([-\sigma, b), E)$ with $\sup_{-\sigma\leq t<b} \|u(t)\| \leq r$, we have $\|(Qu)(t)\| \leq \mu(t)$ for a.e. $t \in [0, b)$;
(h_3) for each bounded set $A \subset \mathcal{C}([-\sigma, b), E)$ we have

$$\alpha((QA)(t)) \leq g(t, \alpha(A(t))) \quad \text{for a.e. } t \in [0, b),$$

where $g : [0, b) \times \mathbb{R}_+ \to \mathbb{R}_+$ is a Kamke function, $A(t) = \{u(t); u \in A\}$ and $(QA)(t) = \{(Qu)(t); u \in A\}$.

By a solution of (1.1) we mean a continuous function $u(\cdot) : [-\sigma, b) \to E$ such that $u|_{[-\sigma,0]} = \varphi$, $u(\cdot)$ is local absolutely continuous on $[0, b)$ and $u'(t) = (Qu)(t)$ for a.e. $t \in [0, b)$.

We remark that $u(\cdot) \in \mathcal{C}([-\sigma, T], E)$, $T > 0$, is a solution for (1.1) on $[-\sigma, T]$ if and only if $u|_{[-\sigma,0]} = \varphi$ and

$$u(t) = \begin{cases} \varphi(t), & \text{for } t \in [-\sigma, 0] \\ \varphi(0) + \displaystyle\int_0^t (Qu)(s)\mathrm{d}s, & \text{for } t \in [0, T]. \end{cases}$$