Nonlinear Analysis 71 (2009) 5897–5905

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Local existence of solutions to dissipative nonlinear evolution equations with mixed types

Hu Wei, Mina Jiang *

Laboratory of Nonlinear Analysis, Department of Mathematics, Huazhong Normal University, Wuhan 430079, PR China

ARTICLE INFO

Article history:
Received 5 June 2007
Accepted 11 March 2009

MSC:
35B40
35F25
35K45

Keywords:
Evolution equation
Local existence
Contraction mapping principle

ABSTRACT

In this paper, we consider the local existence of solutions to the Cauchy problems for the following nonlinear evolution equations with mixed types

$$\begin{cases} \psi_t = -(1-\alpha)\psi - \theta_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\alpha)\theta + \gamma\psi_x + 2\psi\theta_x + \alpha\theta_{xx}, \end{cases}$$

with initial data

$$(\psi, \theta)(x, 0) = (\psi_0(x), \theta_0(x)) \to (\psi_\pm, \theta_\pm), \quad \text{as } x \to \pm\infty,$$

where α and γ are positive constants satisfying $\alpha < 1, \gamma < \alpha(1-\alpha)$. Through constructing an approximation solution sequence, we obtain the local existence by using the contraction mapping principle.

1. Introduction

In physical and mechanical fields, many phenomena can be modeled by systems of nonlinear interaction between ellipticity and dissipation. Due to their complexity, these systems are far from well understood. A set of simplified equations was thus proposed by Hsieh in [1]:

$$\begin{cases} \psi_t = -(\sigma-\alpha)\psi - \sigma\theta_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\beta)\theta + \gamma\psi_x + 2\psi\theta_x + \beta\theta_{xx}, \end{cases} \tag{1.1}$$

where α, β, σ and γ are positive constants satisfying $\alpha < \sigma, \beta < 1$. The complexity of system (1.1) can be explained by a rough argument. If we ignore the damping and diffusion terms temporarily, the system (1.1) becomes:

$$\begin{cases} \psi_t = -\sigma\theta_x, \\ \theta_t = \gamma\psi_x + 2\psi\theta_x. \end{cases} \tag{1.2}$$

We can find that the system (1.2) is elliptic for $|\psi| < \sqrt{\sigma\gamma}$ and hyperbolic, otherwise. Around the zero equilibrium, the system (1.2) subject to initial small disturbances is unstable due to ellipticity, and the inherent instability will cause growth of $|\psi|$ if it overcomes the effect of damping and diffusion terms. But when $|\psi| > \sqrt{\sigma\gamma}$, the system is hyperbolic and ψ ceases to grow. Moreover, the dissipative terms would tend to draw the system back to the elliptic regime; then a "switching back and forth" phenomenon is expected due to the interplay between ellipticity, hyperbolicity and dissipation for suitable coefficients, which makes the system (1.2) quite complicated. There are only a few rigorous results available so far regarding

* Corresponding author.
E-mail address: jmn3911@gmail.com (M. Jiang).

0362-546X/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2009.03.011

this phenomenon, cf. [2–5]. Jian and Chen in [3] first obtained the existence result for the following modified form of the system (1.1)

$$\begin{cases}\psi_t = -(\sigma-\alpha)\psi - \sigma\theta_x + \alpha\psi_{xx},\\ \theta_t = -(1-\beta)\theta + \gamma\psi_x + (\psi\theta)_x + \beta\theta_{xx}.\end{cases} \tag{1.3}$$

Tang and Zhao in [4] discussed the Cauchy problem of the system (1.1) with $\alpha = \beta$ and $\sigma = 1$

$$\begin{cases}\psi_t = -(1-\alpha)\psi - \theta_x + \alpha\psi_{xx},\\ \theta_t = -(1-\alpha)\theta + \gamma\psi_x + 2\psi\theta_x + \alpha\theta_{xx},\end{cases} \tag{1.4}$$

with initial data

$$(\psi,\theta)(x,0) = (\psi_0(x),\theta_0(x)) \to (0,0), \quad \text{as } x \to \pm\infty. \tag{1.5}$$

They obtained the global existence and decay rates of the solutions to the system (1.4)–(1.5) under the assumption $\gamma < 4\alpha(1-\alpha)$ and the initial data

$$(\psi_0(x),\theta_0(x)) \in L^2 \cap W^{1,\infty}(\mathbb{R},\mathbb{R}^2).$$

Zhu and Wang in [5] considered the system (1.4) with more generalized initial data

$$(\psi,\theta)(x,0) = (\psi_0(x),\theta_0(x)) \to (\psi_\pm,\theta_\pm), \quad \text{as } x \to \pm\infty, \tag{1.6}$$

and they also obtained the global existence and decay rates under the assumption of small initial data. Observing the results obtained in [4,5], we can find out that the assumptions imposed on the initial data mean that the system (1.4) is elliptic. But for the system (1.4) with mixed types, i.e., ellipticity and hyperbolicity, the global existence of the solution is an open problem. In order to get the global existence of the solutions to (1.4) with mixed types, obtaining the local existence is the first and essential step, which is the main purpose of this paper. Speaking roughly, we get the local existence of solutions to (1.4) with mixed types by constructing an approximation solution sequence and by the contraction mapping principle in this paper.

Before stating our results precisely, we introduce the following notations.

Notations: Hereafter, we denote several generic positive constants depending on $a, b, \ldots$ by $C_{a,b,\ldots}$ or only by C or $O(1)$ without any confusion and $\delta := |\psi_+ - \psi_-| + |\theta_+ - \theta_-|$. $L^p = L^p(\mathbb{R})(1 \le p \le \infty)$ denotes the usual Lebesgue space with the norm

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}, \quad 1 \le p < \infty,$$
$$\|f\|_{L^\infty} = \sup_{\mathbb{R}} |f(x)|,$$

and the integral region $\mathbb{R}$ will be omitted without any confusion. H^l $(l \ge 0)$ denotes the usual lth-order Sobolev space with the norm

$$\|f\|_{H^l} = \left(\sum_{j=0}^{l} \|\partial_x^j f\|^2\right)^{\frac{1}{2}}.$$

For simplicity, $\|f(\cdot,t)\|_{L^p}$ and $\|f(\cdot,t)\|_{H^l}$ are denoted by $\|f(t)\|_{L^p}$ and $\|f(t)\|_{H^l}$ respectively.

First, we reformulate the system (1.4)–(1.6).

As in [4,5], we introduce the following system

$$\begin{cases}\bar\psi_t = -(1-\alpha)\bar\psi - \bar\theta_x + \alpha\bar\psi_{xx},\\ -(1-\alpha)\bar\theta + \gamma\bar\psi_x = 0,\end{cases} \tag{1.7}$$

or

$$\begin{cases}\bar\psi_t = -(1-\alpha)\bar\psi + \left(\alpha - \dfrac{\gamma}{1-\alpha}\right)\bar\psi_{xx},\\ \bar\theta = \dfrac{\gamma}{1-\alpha}\bar\psi_x,\end{cases} \tag{1.8}$$

where the diffusion equation is obtained by Darcy's law, cf. [6,7].

By direct calculation, the solutions of (1.8) can be written explicitly as

$$\bar\psi(x,t) = \mathrm{e}^{-(1-\alpha)t}\left((\psi_+ - \psi_-)\int_{-\infty}^{x} G(y,t+1)\mathrm{d}y + \psi_-\right), \tag{1.9}$$