Nonlinear Analysis 71 (2009) 5946–5960

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Multiplicity of solutions for elliptic quasilinear equations with critical exponent on compact manifolds

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ARTICLE INFO

Article history:
Received 20 January 2008
Accepted 4 May 2009

MSC:
primary 58J05

Keywords:
Multiplicity of solutions
p-Laplacian operator
Critical Sobolev growth

ABSTRACT

This paper deals with some perturbation of the so-called generalized prescribed scalar curvature type equations on compact Riemannian manifolds; these equations are nonlinear, of critical Sobolev growth, and involve the p-Laplacian. Sufficient conditions are given to have multiple positive solutions.

1. Introduction

Let (M, g) be a compact n-dimensional Riemannian manifold with $n \geq 3$ and k, K, h be real smooth functions on M with $K > 0$. For $p \in (1, n)$, we consider the equation

$$\Delta_p u + k\,|u|^{p-2}\,u = K\,|u|^{p^*-2}\,u + \lambda\,|u|^{q-2}\,u + \epsilon h \tag{1.1}$$

where $\lambda > 0$ is a real parameter and $\epsilon > 0$ is a sufficiently small real number, $\Delta_p u = -div_g(|\nabla u|_g^{p-2}\,\nabla u)$ denotes the p-Laplacian operator, $p^* = \frac{np}{n-p}$ the critical Sobolev exponent and q a real number such that $p < q < p^*$. Let $H_1^p(M)$ be the Sobolev space defined as the completion of $C^\infty(M)$, the space of smooth functions on M, with respect to the norm

$$\|u\|_{1,p} = \|\nabla u\|_p + \|u\|_p\,.$$

Such equation is nonlinear, of degenerated elliptic type, and of critical Sobolev growth.

Motivations for studying equation like (1.1) come from various situations in differential geometry and physics in lacking compactness. A typical example is the Yamabe problem i.e. find a function $u > 0$ on the manifold M such that

$$4\frac{n-1}{n-2}\Delta u + R(x)u = R' u^{\frac{n+2}{n-2}}$$

for some constant R'. $R(x)$ is the scalar curvature.

In [1] Druet studied the existence of positive solutions $u > 0$ to the so-called generalized scalar curvature type equation

$$\Delta_p u + k(x)u^{p-1} = K(x)u^{p^*-1}.$$

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0362-546X/$ – see front matter
doi:10.1016/j.na.2009.05.017

Since p^* is the critical Sobolev growth for which the embedding $H_1^p(M) \hookrightarrow L^{p^*}(M)$ fails to be compact, there are serious difficulties in applying variational methods. Nevertheless, many existence results have been established by solving the subcritical equations and then converging the sequence of subcritical solutions to the critical solution under nonconcentration-compactness conditions. This technique goes to Yamabe himself. In the case of quasilinear equations on manifolds a good reference may be the paper of Druet [1].

The investigation of multiple solutions to semi-linear or quasilinear elliptic equations has drawn the attention of many authors. We quote the work of Ambrossetti–Garcia Azorero–Peral Alonso [2] and Atorero–Peral Alonso [3]. In the context of Riemannian manifolds we mention the works of Hebey–Vaugon [4,5] and also the worthy papers of Rauzy [6,7], where he considered scalar curvature equations with changing right-hand side sign.

In this paper, we investigate existence and multiplicity of positive solutions to (1.1). We show, under some conditions on the operator $Lu = \Delta_p u + k\,|u|^{p-2}\,u$ and on the functions K, h, the existence of at least two positive solutions of (1.1) of class $C^{1,\alpha}(M)$. Firstly, these solutions are obtained without using concentration-compactness methods but for a range of sufficiently large parameters $\lambda > 0$. Secondly, the solutions are established independently of the positive parameter λ via the concentration-compactness techniques. We conclude our paper by a nonexistence result of positive solution to (1.1).

We assume in what follows that the operator $Lu = \Delta_p u + k\,|u|^{p-2}\,u$ is coercive in the sense that there exists $\Lambda > 0$ such that

$$\int_M \left(|\nabla u|^p + k\,|u|^p\right) \mathrm{d}v_g \geq \Lambda\,\|u\|_{1,p}^p\,, \quad u \in H_1^p(M).$$

The main results of this paper state as follows

Theorem 1. *Let (M, g) be a compact Riemannian n-manifold, $n \geq 3$, $p \in (1, n)$ and a real q such that $p < q < p^*$ and k, K, h be smooth real functions on M with*

(i) $K(x) > 0$, $h > 0$, *everywhere on M*

(ii) *the operator $L(u) = \Delta_p u + k(x)\,|u|^{p-2}\,u$ is coercive.*

Then there exist $\lambda_o > 0$ and $\epsilon_o > 0$ such that (1.1) admits, for any $\lambda \geq \lambda_o$ and any $0 < \epsilon \leq \epsilon_o$, at least two distinct positive solutions of class $C^{1,\alpha}$, for some $\alpha \in (0, 1)$.

Theorem 2. *Let (M, g) be a compact Riemannian n-manifold, $n \geq 3$, $p \in (1, n)$ such that $p^2 < n$, $p < q < p^*$ and k, K, h be smooth real functions on M with*

(i) $K(x) > 0$ *and* $h(x) > 0$ *everywhere on M*

(ii) *the operator $L(u) = \Delta_p u + k(x)\,|u|^{p-2}\,u$ is coercive.*

Let x_o be the maximum point of the function K on M i.e. $K(x_o) = \max_{x\in M} K(x)$. Suppose that one of the following conditions are satisfied

(1) $1 < p < 2$, $n - 3p + 2 > 0$ *and* $k(x_o) < 0$.

(2) $p = 2$, $\frac{4(n-1)}{n-2}k(x_o) - Scal(x_o) + \frac{(n-4)}{2}\frac{\Delta K(x_o)}{K(x_o)} < 0$.

(3) $p > 2$, $\frac{\Delta K(x_o)}{K(x_o)} < \frac{p}{n-3p+2} Scal(x_o)$.

Then (1.1) admits, for sufficiently small $\epsilon > 0$, at least two distinct positive solutions of class $C^{1,\alpha}$, for some $\alpha \in (0, 1)$.

1.1. Preliminary results

First we quote a regularity and a strong maximum principle results about equations involving p-Laplacian. The theorems have their origin in Tolksdorf, Gedda-Veron, and Vazquez results in the Euclidean context and have extended to compact manifolds by Druet [1].

Theorem 3 ($C^{1,\alpha}$*-regularity*). *Let (M, g) be a compact Riemannian n-manifold, $n \geq 2$, $p \in (1, n)$, and let $f \in C^o(M \times R)$ be a real function such that*

$$\forall (x, r) \in M \times R, \qquad |f(x, r)| \leq C\,|r|^{p^*-1} + D$$

where C and D are positive constants.

If $u \in H_1^p(M)$ is a weak solution of $\Delta_p u + f(x, u) = 0$

then $u \in C^{1,\alpha}(M)$, for some $0 < \alpha < 1$.

Theorem 4 (*Strong Maximum Principle*). *Let (M, g) be a compact Riemannian n-manifold, $n \geq 2$, $p \in (1, n)$. Let $u \in C^1(M)$ be such that*

$\Delta_p u + f(., u) \geq 0$ on M and the continuous f on $M \times R$ is such that

$$\begin{cases} f(x, r) < f(x, s), & \forall x \in M,\ \forall 0 \leq r \leq s \\ |f(x, r)| \leq C(B + |r|^{p-2})\,|r|\,, & \forall (x, r) \in M \times R \end{cases}$$

where C and B are positive constants.

If $u \geq 0$ on M and u does not vanish identically, then $u > 0$ everywhere on M.