Nonlinear Analysis 71 (2009) 6019–6028

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Strictly localized bounding functions for vector second-order boundary value problems

Jan Andres[a,*], Luisa Malaguti[b], Martina Pavlačková[a]

[a] Department of Math. Analysis and Appl. of Mathematics, Fac. of Science, Palacký University, Tomkova 40, 779 00 Olomouc-Hejčín, Czech Republic

[b] Department of Engineering Sciences and Methods, University of Modena and Reggio Emilia, Via G. Amendola, 2-pad. Morselli, I-42100 Reggio Emilia, Italy

ARTICLE INFO

Article history:
Received 2 April 2009
Accepted 11 May 2009

MSC:
34A60
34B15
47H04

Keywords:
Vector second-order Floquet problem
Strictly localized bounding functions
Solutions in a given set
Scorza–Dragoni technique
Evolution systems
Dry friction problem
Coexistence of periodic and anti-periodic solutions

ABSTRACT

The solvability of the second-order Floquet problem in a given set is established by means of C^2-bounding functions for vector upper-Carathéodory systems. The applied Scorza–Dragoni type technique allows us to impose related conditions strictly on the boundaries of bound sets. An illustrating example is supplied for a dry friction problem.

1. Introduction

Let us consider the Floquet boundary value problem (b.v.p.)

$$\left.\begin{aligned} \ddot{x}(t) + A(t)\dot{x}(t) + B(t)x(t) \in F(t, x(t), \dot{x}(t)), \quad & \text{for a.a. } t \in [0, T], \\ x(T) = Mx(0), \qquad \dot{x}(T) = N\dot{x}(0), \quad & \end{aligned}\right\} \tag{1}$$

where

(1_i) $A, B : [0, T] \to \mathbb{R}^{n\times n}$ are measurable matrix functions such that $|A(t)| \le a(t)$ and $|B(t)| \le b(t)$, for all $t \in [0, T]$ and suitable integrable functions $a, b : [0, T] \to [0, \infty)$,

(1_{ii}) M and N are $n \times n$ matrices, M is nonsingular,

(1_{iii}) $F : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory multivalued mapping.

By a *solution* of problem (1), we mean a vector function $x : [0, T] \to \mathbb{R}^n$ with an absolutely continuous first derivative (i.e. $x \in AC^1([0, T], \mathbb{R}^n)$) which satisfies (1), for almost all $t \in [0, T]$.

* Corresponding author.
E-mail addresses: andres@inf.upol.cz (J. Andres), luisa.malaguti@unimore.it (L. Malaguti), pavlackovam@centrum.cz (M. Pavlačková).

0362-546X/$ – see front matter
doi:10.1016/j.na.2009.05.035

Problem (1) was studied by ourselves via a bound sets approach already in [1]. There the conditions concerning (Lyapunov-like) bounding functions were not imposed directly on the boundaries of bound sets, but at some vicinity of them.

This problem does not occur for Marchaud systems, i.e. for systems with globally upper semicontinuous right-hand sides (see [2]). On the other hand, the case of upper-Carathéodory systems must be furthermore elaborated, for the same goal, by means of suitable Scorza–Dragoni type theorems which is the main aim of this paper.

For the first-order systems, the situation is analogous, but less technical (see [3–7] and cf. [8, Chapter III.8]). Nevertheless, the second-order systems allow us some more flexibility in the sense that the derivatives need not necessarily be taken into account.

The original idea of applying the Scorza–Dragoni technique comes from [9], where guiding functions were employed for vector first-order Carathéodory differential equations. For further references concerning boundary value problems for second-order systems, see, e.g., [1,2,10–12] and the references therein.

Our main result (see Theorem 3.1) shows the solvability of the b.v.p. (1) in the upper-Carathéodory case with strictly localized bounding functions. We separated as much as possible the technicalities needed for its proof into Preliminaries. Its applicability is finally demonstrated by a simple illustrating example for a dry friction problem, when both periodic ($M = N = I$) and anti-periodic ($M = N = -I$) solutions coexist in a given set.

2. Preliminaries

If (X, d) is a metric space and $A \subset X$, by $\overline{A}$ and ∂A, we mean the closure and the boundary of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A : d(x, a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X.

The symbol B_R denotes, as usually, the open ball in $\mathbb{R}^n$ with radius $R > 0$ centered at 0, i.e. $B_R := \{x \in \mathbb{R}^n \mid |x| < R\}$.

Let us recall the following definitions from the multivalued analysis. Let X and Y be arbitrary metric spaces. We say that φ is a multivalued mapping from X to Y (written $\varphi : X \multimap Y$) if, for every $x \in X$, a nonempty subset $\varphi(x)$ of Y is prescribed.

A multivalued mapping $\varphi : X \multimap Y$ is called *upper semicontinuous* (shortly, u.s.c.) if, for each open $U \subset Y$, the set $\{x \in X \mid \varphi(x) \subset U\}$ is open in X.

Let Y be a metric space and $(\Omega, \mathcal{U}, \mu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure μ on $\mathcal{U}$. A multivalued mapping $\varphi : \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid \varphi(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$. In what follows, the symbol μ will exclusively denote the *Lebesgue measure* on $\mathbb{R}$.

We say that mapping $\varphi : J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $\varphi(\cdot, x) : J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $\varphi(t, \cdot) : \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all (a.a.) $t \in J$, and the set $\varphi(t, x)$ is compact and convex, for all $(t, x) \in J \times \mathbb{R}^m$.

Let $X = (X, d)$ be a metric space. A multivalued mapping $\varphi : X \multimap X$ with bounded values is called *Lipschitzian* if there exists a constant $L > 0$ such that

$$d_H(\varphi(x), \varphi(y)) \le L d(x, y),$$

for every $x, y \in X$, where

$$d_H(A, B) := \inf\{r > 0 \mid A \subset N_r(B) \text{ and } B \subset N_r(A)\}$$

stands for the Hausdorff distance. For more information and details concerning multivalued analysis, see, e.g., [8,13–15].

We will also need the following slight modification of the Scorza–Dragoni type result for multivalued mappings.

Proposition 2.1 (*cf., e.g., [16, Proposition 8]*). *Let $X \subset \mathbb{R}^m$ be compact and let $F : [a, b] \times X \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping. Then there exists a multivalued mapping $F_0 : [a, b] \times X \multimap \mathbb{R}^n \cup \{\emptyset\}$ with compact, convex values and $F_0(t, x) \subset F(t, x)$, for all $(t, x) \in [a, b] \times X$, having the following properties:*

(i) *if $u, v : [a, b] \to \mathbb{R}^n$ are measurable functions with $v(t) \in F(t, u(t))$, on $[a, b]$, then $v(t) \in F_0(t, u(t))$, a.e. on $[a, b]$;*
(ii) *for every $\varepsilon > 0$, there exists a closed $I_\varepsilon \subset [a, b]$ such that $\mu([a, b] \setminus I_\varepsilon) < \varepsilon$, $F_0(t, x) \neq \emptyset$, for all $(t, x) \in I_\varepsilon \times X$, and F_0 is u.s.c. on $I_\varepsilon \times X$.*

It will be convenient to recall some basic facts concerning evolution systems. For a suitable introduction and further details, we refer, e.g., to [17].

Hence, let $C : [a, b] \to \mathbb{R}^{m\times m}$ be a measurable matrix function such that $|C(t)| \le c(t)$, for all $t \in [a, b]$, with $c \in L^1([a, b], [0, \infty))$ and let $f \in L^1([a, b], \mathbb{R}^m)$. Given $x_0 \in \mathbb{R}^m$, consider the linear initial value problem

$$\dot{x}(t) = C(t)x(t) + f(t), \quad x(a) = x_0. \tag{2}$$

It is well-known (see, e.g., [17]) that, for the uniquely solvable problem (2), there exists the evolution operator $\{U(t, s)\}_{(t,s)\in\Delta}$, where $\Delta := \{(t, s) : a \le s \le t \le b\}$, such that

$$|U(t, s)| \le e^{\int_s^t |C(\tau)|\, d\tau}, \quad \text{for all } (t, s) \in \Delta; \tag{3}$$