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On ground state solutions for singular and semi-linear problems including super-linear terms at infinity

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ABSTRACT

We establish a result concerning the existence of entire, positive, classical and bounded solutions which converge to zero at infinity for the semi-linear equation $-\Delta u = \lambda f(x, u)$, $x \in \mathbb{R}^N$, where $f : \mathbb{R}^N \times (0, \infty) \to [0, \infty)$ is a suitable function and $\lambda > 0$ is a real parameter. This result completes the principal theorem of A. Mohammed [A. Mohammed, Ground state solutions for singular semi-linear elliptic equations, Nonlinear Analysis (2008) doi:10.1016/j.na.2008.11.080] mainly because his result does not address the super-linear terms at infinity. Penalty arguments, lower–upper solutions and an approximation procedure will be used.

1. Introduction

In the present paper we establish a new result concerning the existence of solution for the nonlinear semi-linear problem

$$\begin{cases} -\Delta u = \lambda f(x, u) & \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, & u(x) \to 0, \quad |x| \to \infty, \end{cases} \tag{1.1}$$

where $f : \mathbb{R}^N \times (0, \infty) \to [0, \infty)$ is a continuous function and $\lambda > 0$ is a real parameter.

The class of problems (1.1) appears in many nonlinear phenomena, for instance, in the theory of quasi-regular and quasi-conformal mappings [1–3], in the generalized reaction–diffusion theory [4], in the turbulent flow of a gas in porous medium and in the Newtonian fluid theory [5].

It follows by the non-negativity of the function f, of parameter λ and a strong maximum principle that all non-negative and non-trivial solutions of (1.1) must be strictly positive (see Serrin and Zou [6]). So, again from [6], it follows that (1.1) admits solutions if and only if $N \geq 3$.

The particular case $f(x, u) = b(x)g(u)$, where $b : \mathbb{R}^N \to (0, \infty)$ is locally Hölder continuous and $g : (0, \infty) \to (0, \infty)$ is a C^1 suitable function has been considered in (1.1) in recent years by a number of authors. More specifically, the problem:

$$\begin{cases} -\Delta u = \lambda b(x)g(u) & \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, & u(x) \to 0, \quad |x| \to \infty. \end{cases} \tag{1.2}$$

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Consider the hypotheses under b:
(b_1) $\int_1^\infty t\hat{b}(t)\mathrm{d}t < \infty$, where $\hat{b}(t) = \max_{|x|=t} b(x)$, for $t > 0$,
(b_2) the problem

$$\begin{cases} -\Delta u = b(x) & \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N, & u(x) \to 0, \quad |x| \to \infty \end{cases} \tag{1.3}$$

has a unique solution $w_b \in C^{2,\alpha}_{loc}(\mathbb{R}^N)$.

Also concerning g:

(g_1) g is non-increasing, (g_2) $\lim_{t\to 0} g(t) = \infty$,
(g_3) g is bounded in a neighborhood at ∞, (g_4) $\lim_{t\to 0} \frac{g(t)}{t} = \infty$,
(g_5) $\frac{g(t)}{t+c}$ is decreasing for some $c > 0$, (g_6) $\frac{g(t)}{t}$ is decreasing,
(g_7) $\limsup_{t\to\infty} \frac{g(t)}{t} < \frac{1}{\|w_b\|_\infty}$, ($g_8$) $\lim_{t\to\infty} \frac{g(t)}{t} = 0$.

A brief overview. The problem (1.2), with $\lambda = 1$, has been studied and ground state solutions have been established by a number of authors. Lair and Shaker [7] and Zhang [8] proved that the problem (1.2) has a solution under (g_1) and (g_2) hypotheses. Feng and Liu in [9] established a solution for (1.2) if (g_2) and (g_3) hold. Cîrstea and Rădulescu in [10] showed that (1.2) has a unique ground state if (g_3), (g_4) and (g_5) are satisfied. Goncalves and Santos in [11] improved the earlier results because under assumptions (g_4), (g_6) and (g_8) proved the existence of solutions for (1.2). Zhang in [12] improved all earlier results because he assumed only (g_4), (g_8) and non-monotonicity condition was required.

To complete the overview, first we consider the following eigenvalue problem

$$\begin{cases} -\Delta u = \lambda\rho(x)u & \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, & u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ and $\rho \neq 0$ is a non-negative function locally Hölder continuous.

It is known that the first eigenvalue variational $\lambda_1 = \lambda_1(\rho, \Omega)$ satisfies $\lambda_1(\rho, \Omega_1) \geq \lambda_1(\rho, \Omega_2)$ if $\Omega_1 \subseteq \Omega_2$. So there exists

$$\lambda_1(\rho) = \lim_{k\to\infty} \lambda_1(\rho, B_k(0)) \in [0, \infty), \tag{1.5}$$

where $B_k(0)$ is the ball centered at the origin and radius $k = 1, 2, \ldots$. Additionally, $\lambda_1(\rho)$ is positive if (b_1) is satisfied with ρ in the place of b.

Now, I suppose

(f_1) $f(x, s)$ is locally Hölder continuous em $\mathbb{R}^N \times (0, \infty)$ and continuous differentiable in the variable s,
(f_2) $f(x, s) \leq b(x)g(s)$ for all $(x, s) \in \mathbb{R}^N \times (0, \infty)$ for some $b : \mathbb{R}^N \to [0, \infty)$, $b \neq 0$ and $g : (0, \infty) \to (0, \infty)$ continuous functions,
(f_3) $f(x, s) \geq a(x)h(s)$ for all $(x, s) \in \mathbb{R}^N \times (0, s_0)$ for some $0 < s_0 \leq 1$, $a : \mathbb{R}^N \to [0, \infty)$, $a \neq 0$ and $h : (0, s_0) \to (0, \infty)$ continuous functions.

In a recent result A. Mohammed [13] improved all earlier results about the existence of ground state solutions for problems like (1.1), which includes (1.2) as a particular case, because he proved the existence under (f_1), (f_2) and (f_3) hypotheses with a and b strictly positive, b satisfying (b_2), $h(s) = \lambda s$, $s > 0$ with $\lambda > \lambda_1(a, B_1(0))$ and g as in (g_7).

To state our main result we will establish the following notations

$$h_0 = \lim_{s\to 0} \frac{h(s)}{s} \in (0, \infty] \quad \text{and} \quad g_\infty = \lim_{s\to\infty} \frac{g(s)}{s} \in [0, \infty].$$

Theorem 1.1. *Suppose* (f_1)–(f_3) *with b satisfying* (b_2) *hold. Then the problem* (1.1) *has a solution* $u = u_\lambda \in C^2(\mathbb{R}^N)$ *if either*

(i) $g_\infty < \infty$ *and* $\lambda_1(a)/h_0 < \lambda < 1/g_\infty\|w_b\|_\infty$, *or*
(ii) $g_\infty = \infty$, $h_0 = \infty$ *and* $0 < \lambda < \lambda^\star$, *for some* $\lambda^\star > 0$.

Additionally, if (b_1) *holds then there exist* $c, d > 0$ *constants such that*

$$c|x|^{2-N} \leq u(x) \leq d|x|^{2-N}, \quad |x| \geq 1.$$

Theorem 1.1 completes the principal result of Mohammed in [13] not only because it permits super-linear nonlinearities at infinity but also because the terms a and b are not assumed to be positive throughout $\mathbb{R}^N$.