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Strong convergence theorems of modified Ishikawa iterative process with errors for an infinite family of strict pseudo-contractions☆

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ABSTRACT

In this paper, we study modified Ishikawa iterative process with errors to have strong convergence for an infinite family of strict pseudo-contractions in the framework of q-uniformly smooth Banach spaces. Our results improve and extend the recent ones announced by many others.

1. Introduction

Throughout this paper, we denote by E and E^* a real Banach space and the dual space of E, respectively. Let C be a subset of E and T be a self-mapping of C. We use $F(T)$ to denote the fixed points of T. Let $q > 1$ be a real number. The (generalized) duality mapping $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \left\{x^* \in E^* : \langle x, x^*\rangle = \|x\|^q, \left\|x^*\right\| = \|x\|^{q-1}\right\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2} J_2(x)$ for $x \neq 0$. If E is a Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if E is smooth, then J_q is single-valued, which is denoted by j_q.

Recall that T is a nonexpansive mapping if

$$\|Tx - Ty\| \le \|x - y\|, \tag{1.1}$$

for all $x, y \in C$.

A mapping T is called a pseudo-contraction, if there exists some $j_q(x - y) \in J_q(x - y)$ such that

$$\left\langle Tx - Ty, j_q(x - y)\right\rangle \le \|x - y\|^q, \tag{1.2}$$

for all $x, y \in C$.

T is said to be a λ-strict pseudo-contraction in the terminology of Browder and Petryshyn [1], if there exists a constant $\lambda > 0$ such that

$$\left\langle Tx - Ty, j_q(x - y)\right\rangle \le \|x - y\|^q - \lambda \|(I - T)x - (I - T)y\|^q, \tag{1.3}$$

for every $x, y \in C$ and for some $j_q(x - y) \in J_q(x - y)$.

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Let C be a nonempty closed convex subset of a real Hilbert space H, and $T : C \to C$ be a mapping. T is said to be a k-strict pseudo-contraction in the sense of Browder and Petryshyn [1], if there exists a $\kappa \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \tag{1.4}$$

for every $x, y \in C$.

T is said to be a strong pseudo-contraction if there exists $\kappa \in (0, 1)$ such that

$$\langle Tx - Ty, j_q(x - y)\rangle \le \kappa \|x - y\|^q, \tag{1.5}$$

for every $x, y \in C$.

Remark 1.1. From (1.3) we can prove that if T is λ-strict pseudo-contractive, then T is Lipschitz continuous with the Lipschitz constant $L = \frac{1+\lambda}{\lambda}$. The class of strongly pseudo-contractive mappings is independent of the class of λ-strict pseudo-contractions (see, e. g., Zhou [2]).

A self-mapping $f : C \longrightarrow C$ is a contraction on C, if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \le \alpha \|x - y\|, \quad \forall x, y \in C.$$

We use $\prod_C$ to denote the collection of all contractions on C. That is, $\prod_C = \{f | f : C \to C \text{ a contraction}\}$.

Let $S(E) = \{x \in E : \|x\| = 1\}$. Then the norm of E is said to be Gâteaux differentiable if

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{Δ}$$

exists for each $x, y \in S(E)$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if, for each $y \in S(E)$, the limit (Δ) is attained uniformly for $x \in S(E)$. The norm of the E is said to be Frêchet differentiable, if for each $x \in S(E)$, the limit (Δ) is attained uniformly for $y \in S(E)$. The norm of E is called uniformly Frêchet differentiable, if the limit (Δ) is attained uniformly for $x, y \in S(E)$. It is well known that (uniform) Frêchet differentiability of the norm of E implies (uniform) Gâteaux differentiability of norm of E.

A Banach space E is said to be strictly convex if, whenever x and y are not colinear, $\|x + y\| < \|x\| + \|y\|$.

Let $\rho_E : [0, \infty) \longrightarrow [0, \infty)$ be the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : x \in S(E),\ \|y\| \le t\right\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. A Banach space E is said to be q-uniformly smooth, if there exists a fixed constant $c > 0$ such that $\rho_E(t) \le ct^q$. It is well known that E is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth, then $q \le 2$ and E is uniformly smooth, and hence the norm of E is uniformly Fréchet differentiable, in particular, the norm of E is Fréchet differentiable. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$.

Recall that, if C and D are nonempty subsets of a Banach space E such that C is nonempty closed convex and $D \subset C$, then a mapping $Q : C \longrightarrow D$ is sunny [3] provided

$$Q(x + t(x - Q(x))) = Q(x) \quad \text{for all } x \in C \text{ and } t \ge 0,$$

whenever $x + t(x - Q(x)) \in C$. A sunny nonexpansive retraction is a sunny retraction, which is also a nonexpansive mapping.

In a real q-uniformly smooth Banach space, Xu [4] proved the important inequality.

Lemma 1.2. *Let E be a real q-uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x + y\|^q \le \|x\|^q + q\langle y, j_q x\rangle + C_q \|y\|^q,$$

for all $x, y \in E$. In particular, if E is real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, jx\rangle + 2\|Ky\|^2,$$

for all $x, y \in E$.

The relation between the λ-strict pseudo-contractive mapping and the nonexpansive mapping can be obtained from the following Lemma.

Lemma 1.3 ([5]). *Let C be a nonempty convex subset of a real q-uniformly smooth Banach space E and $T : C \to C$ be a λ-strict pseudo-contraction. For $\alpha \in (0, 1)$, we define $T_\alpha x = (1-\alpha)x + \alpha Tx$. Then, as $\alpha \in (0, \mu]$, $\mu = \min\left\{1, \left\{\frac{q\lambda}{C_q}\right\}^{\frac{1}{q-1}}\right\}$, $T_\alpha : C \to C$ is nonexpansive such that $F(T_\alpha) = F(T)$.*