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Fixed point results for generalized quasicontraction mappings in abstract metric spaces☆

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ABSTRACT

In this paper, we introduce the concept of generalized quasicontraction mappings in an abstract metric space. By using this concept, we construct an iterative process which converges to a unique fixed point of these mappings. The result presented in this paper generalizes the Banach contraction principle in the setting of metric space and a recent result of Huang–Zhang for contractions. We also validate our main result by an example.

1. Introduction

In the 60s of the last century, the notion of a metric space was generalized, replacing the set of real numbers by an ordered linear space with a cone K and the new notion was called, a K-metric space or a generalized metric space. Since then, this notion was used for establishing fixed point theorems, see, e.g., [1–10] and also the references therein. Recently, Huang–Zhang [11] rediscovered this notion and called it a cone metric space (It seems that they were unaware of the earlier work.). Huang–Zhang also obtained some fixed point theorems of contractive mappings in cone metric spaces or P-metric spaces. One of their main results is the following:

Theorem 1.1. *Let (M, d) be a complete P-metric space and P be a normal cone with normal constant K. Suppose that the mapping $T : M \to M$ satisfies the contractive condition*

$$d(Tx, Ty) \preceq \lambda \cdot d(x, y) \tag{1.1}$$

for all $x, y \in M$, where $\lambda \in [0, 1)$ is a constant. Then T has a unique fixed point in M and for any $x \in M$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Let (M, ρ) be a P-metric space. A map $T : M \to M$ such that for some constant $\lambda \in (0, 1)$ and for every $x, y \in M$, there exists $u \in C(T, x, y) := \{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \rho(x, Ty), \rho(y, Tx)\}$, such that

$$\rho(Tx, Ty) \preceq \lambda u \tag{1.2}$$

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is called *quasicontraction* (see [12]). Ciric [13,14] introduced quasicontraction in metric spaces as one of the most general contractive type map and proved that every quasicontraction map T possesses a unique fixed point in a complete metric space.

The aim of this paper is to introduce the concept of generalized quasicontraction mappings in P-metric space. We construct an iterative process which converges to a unique fixed point of these mappings in cone metric spaces. The result presented in this paper generalizes the Banach contraction principle in the setting of metric space and a recent result of Huang–Zhang for contractions in complete P-metric spaces. We also validate our main result by an example.

2. Preliminaries

First of all we review some basic definitions and notations. Let E be always a real Banach space and P be a subset of E.

Definition 2.1. P is called a cone if and only if

(1) P is closed, nonempty and $P \neq \{0\}$;
(2) if $a, b \in R$, $a, b \geq 0$ then $x, y \in P \Rightarrow ax + by \in P$;
(3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. If $P \subset E$ is endowed with partial ordering $\preceq$ as defined above, then the pair $(E, \preceq)$ is called an ordered Banach space and the cone P is called a positive cone. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, int$P$ denotes the interior of P. The cone P is called normal if $\inf\{\|x + y\| : x, y \in P \cap \partial B_1\} > 0$. The norm $\|\cdot\|$ on E is called *semimonotone* if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \preceq x \preceq y \quad \text{implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above relation is called the *normal constant* of P. It may be remarked that P is normal iff $\|\cdot\|$ is semimonotone. The norm $\|\cdot\|$ on E is called *monotone* if $0 \preceq x \preceq y$ implies $\|x\| \leq \|y\|$. The cone P is called *regular* if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is a sequence such that $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \cdots \preceq y$ (or $y \preceq \cdots \preceq x_n \preceq x_{n-1} \preceq \cdots \preceq x_2 \preceq x_1$) for some $y \in E$, then there is a $x \in E$ such that $\|x_n - x\| \to 0$, $n \to \infty$. Equivalently, the cone P is regular if and only if every increasing (resp. decreasing) sequence which is bounded from above (resp. below) is convergent. It is well known that a regular cone is a normal cone.

Definition 2.2. A map $f : X \to E$ is called *T-orbitally lower semicontinuous at x*, if for any sequence $\{T^n x\}$ in X and $\bar{x} \in X$ such that $T^n x \to \bar{x}$, we have $f(\bar{x}) \leq \liminf_{n\to\infty} f(T^n x)$.

Definition 2.3. Let M be a nonempty set. Suppose the mapping $d : M \times M \to E$ satisfies

(d1) $0 \preceq d(x, y)$ for all $x, y \in M$ and $d(x, y) = 0$ if and only if $x = y$;
(d2) $d(x, y) = d(y, x)$ for all $x, y \in M$;
(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in M$.

Then d is called a *cone metric* on M, and (M, d) is called a *cone metric space* or a *P-metric space*.

Let (M, d) be a P-metric space. Let $\{x_n\}$ be a sequence in M. We say that $\{x_n\}$ is convergent to some $x \in M$, if for any $c \in E$ with $0 \ll c$ there exists N such that for all $n > N$, $d(x_n, x) \ll c$. We denote this by $\lim_{n\to\infty} x_n = x$. We say that $\{x_n\}$ is a Cauchy sequence in M if for any $c \in E$ with $0 \ll c$ there exists N such that for all $n, m > N$, $d(x_n, x_m) \ll c$.

A space M is said to be a complete P-metric space if every Cauchy sequence is convergent in M. If $\{x_n\}$ is convergent to some $x \in M$, then $\{x_n\}$ is a Cauchy sequence. If P is a normal cone with normal constant K then: (i) $\{x_n\}$ converges to x iff $\lim_{n\to\infty} d(x_n, x) = 0$; (ii) $\{x_n\}$ is a Cauchy sequence iff $\lim_{n,m\to\infty} d(x_n, x_m) = 0$; (iii) if $\{x_n\}$ and $\{y_n\}$ are two sequences in M such that $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ for some $x, y \in M$, then $\lim_{n,m\to\infty} d(x_n, y_n) = d(x, y)$.

Let $T : M \to M$ and let $O(x; \infty)$ be an orbit of T at a point $x \in M$ i.e., the set of the form $\{x, Tx, T^2x, \ldots, T^n x, \ldots\}$. A space M is said to be T-orbitally complete if every sequence $\{T^{n_i} x\}_{i\in\mathbb{N}}$, $x \in M$, which is a Cauchy sequence, has a limit point in M. If M is a complete space, then M is T-orbitally complete with respect to any self-mapping T on M.

Let (M, ρ) be a metric space. A mapping $T : M \to M$ is said to be a Banach contraction mapping if there exists $0 < \lambda < 1$ such that $\rho(Tx, Ty) \leq \lambda\rho(x, y)$ for all x, y in M. It is obvious that the Banach contraction mapping is continuous and it is well known that in complete metric spaces it has a unique fixed point [15]. In [16] the concept of p-contraction mappings was introduced. A mapping $T : Y \subset M \to M$ is said to be a p-contraction mapping if Y is T-invariant and it satisfies the following inequality

$$\rho(Tx, T^2x) \leq p(x)\rho(x, Tx) \quad \text{for all } x \text{ in } Y, \tag{2.1}$$

where $p : Y \to [0, 1)$ is a mapping such that $\sup_{x\in Y} p(Tx) = \lambda < 1$. Further, if $\bigcap_{n=0}^{\infty} T^n(Y)$ is a singleton set, where $T^n(Y) := TT^{n-1}(Y)$ for each $n \in \mathbb{N}$ and $T^0(Y) := Y$, then T is said to be a strong p-contraction.

Examples 2.1 and 2.2 in [16] show that the p-contraction mapping is essentially more general than the Banach contraction mapping and not necessarily need to be continuous.