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KKM–Fan's lemma for solving nonlinear vector evolution equations with nonmonotonic perturbations

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ABSTRACT

In this paper we are interested in the existence of solutions of the following initial value problem: $\mathrm{d}u/\mathrm{d}t + Au + Gu = f$ on $(0, T)$ with $u(0) = u_0$ where $A : V \to V'$ is a monotone operator, $G : V \to V'$ is a nonlinear nonmonotone operator and $f : (0, T) \to V'$ is a measurable function, by means of a recent generalization of the famous KKM–Fan's lemma.

1. Introduction and preliminaries

Let V be a reflexive Banach space which is supposed to be densely and continuously imbedded in a real Hilbert space H. In this paper we are interested in the existence of solutions of the following mixed initial vector value problem:

$$(\text{MIVV}) \begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}u + Au + Gu = f \quad \text{on } (0, T) \\ u(0) = u_0 \end{cases}$$

where $A : V \to V'$ is a monotone operator, $G : V \to V'$ is neither linear nor monotone and $f : (0, T) \to V'$ is a measurable function. Recently, (MIVV) has been studied by many authors. For a brief history of problems of this kind, following [1], one can cite: If $G = 0$ we are brought back to the theory of maximal monotone operators see [2]. Browder [3], Pavel [4], Pavel and Vrabie [5] studied the problem in the case where A is linear. The nonlinear case was studied by Gutman [6], Hirano [1], Vrabie [7], Ahmed and Xiang [8], Zhenhai [9] and many authors.

Our purpose in this paper is to extend the results of Hirano [1], Ahmed and Xiang [8] and Zhenhai [9]. The first approach, adopted in [1] and [8], reduces (MIVV) to the equivalent functional equation

$$L^*v + L^*[A(Lv + u_0) - f + G(Lv + u_0)] = 0 \tag{1}$$

where $v = \frac{\mathrm{d}}{\mathrm{d}t}u$, $Lv(t) = \int_0^t v(s)\mathrm{d}s$ and L^* is the adjoint operator of L, and uses the theory, see [3], of pseudomonotone operators to $L^* + L^*[A(L + u_0) - f + G(L + u_0)]$. The second one [9] applies Browder's existence result, see Zeidler [2], of

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zero for the sum of the pseudomonotone operator $Pu = A(u - u_0) - f + G(u - u_0)$ and the maximal monotone operator $Mu = \frac{d}{dt}u$ where M is defined on $D(M) = \{u \in X : \frac{d}{dt}u \in X', u(0) = u_0\}$ and $p \geq 2$ with $1/p + 1/q = 1$. Our goal in this paper is to adopt the first approach and solve (MIVV) by means of a recent generalization of the famous KKM–Fan's lemma, see [10]. Hirano [1, Theorems 1 and 2] sets the range of G to be included in H and the imbedding $V \hookrightarrow H$ to be compact. Ahmed and Xiang [8, Theorem B] extended Hirano's result in replacing compactness inclusion of V in H by $\langle Gx_n, x_n - x\rangle \to 0$ if $\{x_n\}$ converges weakly to x. Zhenhai [9, Theorem 2] extends the next condition by the following broader one: $\liminf\langle Gx_n, x_n - x\rangle \geq 0$ if $\{x_n\}$ converges weakly to x but setting A to be demicontinuous instead of hemicontinuous. As explained in Remark 2.2, Theorem 2.1 extends all the results cited above by coming back to hemicontinuity and replacing $\langle Gx, x\rangle \geq -c$ for some positive c by $\langle Gx, x\rangle \geq g(x)$ for some weakly continuous real function $g : V \longrightarrow \mathbb{R}$ which satisfies $g(x) \leq \eta\left(\|x\|^{\delta} + 1\right)$ where η and δ are positive constants.

2. The main result

Let us first describe briefly the set of general data and assumptions imposed on our model of nonlinear nonmonotone initial value problem. We consider throughout, unless explicitly stated otherwise, the case of a reflexive Banach space V which is supposed to be densely and continuously imbedded in a real Hilbert space H. The norm of V and H are denoted by $\|.\|_V$ and $\|.\|_H$. Identifying H with its dual, we have that $V \subset H \subset V'$, and if $\langle x, y\rangle$ denotes the duality pairing between $x \in V$ and $y \in V'$ then for $x, y \in H$, $\langle x, y\rangle$ is the ordinary inner product of H.

An operator $B : V \to V'$ is said to be monotone if for each $x, y \in V$

$$\langle Bx - By, x - y\rangle \geq 0.$$

B is called pseudomonotone if $\{x_n\}$ weakly converges to x and

$$\limsup \langle Bx_n, x_n - x\rangle \leq 0,$$

implies that for each $y \in V$

$$\langle Bx, x - y\rangle \leq \liminf \langle Bx_n, x_n - y\rangle .$$

We say that B is hemicontinuous if for each $x, y, z \in V$ the function $s \mapsto \langle A(x + sy), z\rangle$ is weakly continuous at $s = 0$.

Let p, q and T be constants such that $T > 0, p \geq 2$ and $1/p + 1/q = 1$, then by $L^p(0, T; V)$ we denote the space of strongly measurable V-valued functions such that $\int_0^T \|u(t)\|_V^p dt < \infty$. Likewise we denote by $L^q(0, T; V')$ the space of strongly measurable V'-valued functions v such that $\int_0^T \|v(t)\|_{V'}^q dt < \infty$. The pairing between $X = L^p(0, T; V)$ and $X' = L^q(0, T; V')$ is denoted by $\langle\langle ., .\rangle\rangle$. We denote by J the duality mapping from X' into X, i.e. for each $v \in X', J(v) =: \left\{u \in X : \langle\langle u, v\rangle\rangle = \|u\|_X^2 = \|v\|_{X'}^2\right\}$.

We will also denote by L the linear operator $(Lu)(t) = \int_0^t u(s)ds$ and the associated adjoint operator $(L^*u)(t) = \int_t^T u(s)ds$.

In the following we will suppose that the operators $A, G : V \longrightarrow V'$ satisfy the following basic assumptions:

(A_1) A is monotone and hemicontinuous.

(A_2) There exist positive constants c_1, c_2 and c_3 such that for each $x \in V$

$$\|Ax\|_{V'} \leq c_1(\|x\|_V^{p-1} + 1) \quad \text{and} \quad c_2 \|x\|_V^p \leq c_3 + \langle Ax, x\rangle .$$

(G_1) G is both continuous and weakly continuous.

(G_2) There exist positive constants a, b such that for each $x \in V$

$$\|Gx\|_{V'} \leq a \|x\|_V^{p-1} + b.$$

(G_3) There exist a weakly continuous real function $g : V \longrightarrow \mathbb{R}$ and positive constants η and δ such that for each $x \in V$

$$g(x) \leq \eta\left(\|x\|^{\delta} + 1\right) \quad \text{and} \quad \langle Gx, x\rangle \geq -g(x).$$

(G_4) If $\{x_n\}$ weakly converges to x in V then we have

$$\limsup \langle Gx_n, x - x_n\rangle \leq 0.$$

Theorem 2.1. *Suppose that* (A_1), (A_2), (G_1), (G_2), (G_3) *and* (G_4) *hold. Then for each* $u_0 \in V$ *and* $f \in X'$, *there exists a solution* u *of (MIVV) such that* $u \in C(0, T; H) \cap X$ *and* $\frac{d}{dt}u \in X'$.

Before proving this result we need to make a few comments on assumptions.

Remark 2.2. (1) First note that [9, Lemma 2] needs "$\{x_n\}$ converges weakly to x implies $\liminf \langle Gx_n, x_n - x\rangle \geq 0$", which is equivalent to our assumption (G_4), than the imposed condition: "$\{x_n\}$ converges weakly to x implies

$$\limsup \langle Gx_n, x_n - x\rangle \geq 0".$$

(2) Consider $V = H = \mathbb{R}$, $k, k' \in \mathbb{N}$ and the mappings A, G, g defined on $\mathbb{R}$ by $Ax = x^{2k-1}$, $Gx = -x^{2k'-2}$ and $g(x) = x^{2k'-1}$ where $1 < k' < k$.

Set $p = 2k$ and suppose that $\eta, c_3, b > 1$ and $2k' - 1 < \delta < p$. Then A is monotone but G is not. Remark that all conditions (A_1), (A_2), (G_1)–(G_4) are satisfied, however the imposed condition in [1,8,9], which means "$\langle Gx, x\rangle \geq -c$ for some positive constant c" is not satisfied.