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Existence and uniqueness of solutions of piecewise nonlinear systems[☆]

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ABSTRACT

In this paper, we consider the existence and the uniqueness of solutions of piecewise nonlinear systems. We will present some necessary and sufficient conditions for the existence and the uniqueness of solutions of this class of systems.

1. Introduction

Piecewise system is an important class of hybrid systems. The study of this class of systems dates back to the early works of Andronov on oscillation in nonlinear systems, Kalman on saturated linear systems in 1950s. This field has received much attention over the past two decades [1–12]. However, most of these works are based on the assumption that this class of systems is well-posed.

In [11], the authors have studied the well posedness of piecewise linear systems in the sense of Carathéodory. They first used the lexicographic inequalities and the smooth continuation to derive necessary and sufficient conditions for the well posedness of a bimodal system with single criterion. They then extended those results to a multi-modal system with multiple criteria. In [11], an algorithm is proposed for solving these conditions. In [12], some sufficient conditions are obtained for the well posedness of the switch based control systems. In [13], necessary and sufficient conditions for the well posedness of piecewise linear systems with multiple modes and multiple criteria are derived. A computational procedure is then developed to solve these necessary and sufficient conditions by using the Fourier–Motzkin elimination rather than the linear programming method as in [11]. However, all the above results are for linear systems. It appears that few results are available for cases involving nonlinear systems. In this paper, we study the existence and uniqueness of solutions of piecewise nonlinear systems. Some necessary and sufficient conditions for this class of systems will be derived. Our results generalize those obtained in [11,13].

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The organization of the paper is as follows. In Section 2, we formulate the problem. Section 3 is devoted to developing necessary and sufficient conditions. Some concluding remarks are given in Section 4.

In what follows, we will use the lexicographic inequalities of $x \in \mathbb{R}^n$, i.e.,

$$x \succeq 0 \Leftrightarrow \text{for some } i, \quad x_j = 0 \quad (j = 1, 2, \ldots, i-1), \quad x_i > 0 \quad \text{or } x = 0.$$

2. Piecewise nonlinear system with Multi-modal and multi-criteria

First, let us carefully examine the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} f_1(x_1, x_2), & \text{if } x_2 - x_1^2 \geq 0, \\ f_2(x_1, x_2), & \text{if } x_2 - x_1^2 \leq 0, x_2 + x_1^2 \geq 0, x_1 \leq 0, \\ f_3(x_1, x_2), & \text{if } x_2 - x_1^2 \leq 0, x_2 + x_1^2 \geq 0, x_1 \geq 0, \\ f_4(x_1, x_2), & \text{if } x_2 + x_1^2 \leq 0. \end{cases} \tag{1}$$

Here, $\mathbb{R}^2$ is partitioned into four parts R^i, $i = 1, \ldots, 4$, which is shown in Fig. 1. In each R^i, the system is evolved according to the model $\dot{x} = f_i(x)$.

A question emerges naturally: when does the system (1) admit a unique solution in some sense for any initial condition $x_0 \in \mathbb{R}^2$.

To address this question, let us consider a general system given by

$$\Sigma : \dot{x}(t) = f_i(x), \quad \text{if } y = \left[h_i^1, \ h_i^2, \ldots, h_i^{p_i}(x)\right] \succeq 0, \tag{2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $f_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, \ldots, m$, and $h_i^j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, p_i$, are real functions defined on $\mathbb{R}^n$.

When $m = 2$, the system (2) is called bimodal. For a bimodal system, if $p_1 = p_2 = 1$, then the system is called a bimodal system with a single criterion.

Suppose that $x(t)$ satisfies $h_i^j(x) = 0, j = 1, \ldots, p_i$, and $h_k^l(x) = 0$, $l = 1, \ldots, p_k$, at some time $\hat{t}$. Then, which mode is to be applied is determined by the behavior of $x(t)$ in the time interval $\left[\hat{t}, \hat{t} + \varepsilon\right]$, where $\varepsilon > 0$ is a small constant.

Let us first recall some definitions given in [11].

Definition 2.1. For a given initial state $x(t_0)$, suppose that $x(t)$ satisfies

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau))\, \mathrm{d}\tau \tag{3}$$

and is absolutely continuous on each compact subinterval of $[t_0, \ t_1)$, where $f(x)$ is the vector field given by the right hand of (2). If there exists no left-accumulation point [11] of event times on $[t_0, \ t_1)$, then $x(t)$ is said to be a solution of system (2) on $[t_0, \ t_1)$ in the sense of Carathéodory for the initial state $x(t_0)$.

Definition 2.2. Let $\mathcal{S}$ be a subset of $\mathbb{R}^n$. If for a given initial state x_0, there exists an $\varepsilon > 0$ such that for all $t \in [0, \varepsilon]$, $x(t) \in \mathcal{S}$, then we say that the system has the smooth continuation property at x_0 with respect to $\mathcal{S}$. Moreover, if from any $x_0 \in \mathcal{S}$, the smooth continuation is possible with respect to $\mathcal{S}$, then the system is said to have the smooth continuation property with respect to $\mathcal{S}$.

Throughout the paper, we assume that the following conditions are satisfied:

Assumption A. (1) $f_i, h_i^j, i = 1, \ldots, m; j = 1, \ldots, p_i$, are analytic functions.

(2) For each $i = 1, \ldots, m$, and any $M > 0$, there exists a $K_{i,M}$ such that

$$\|f_i(x)\| \leq K_{i,M}(1 + \|x\|), \text{ for any } x \in \left\{x \in \mathbb{R}^n : \|x\| \leq M\right\}, \tag{4}$$

where $\|\cdot\|$ denotes the usual norm of $\mathbb{R}^n$.

3. Main results

In this section, we will give some necessary and sufficient conditions for the existence and the uniqueness of solutions of system (2).

Lemma 3.1. *If f_i, $i = 1, \ldots, m$, satisfy (4), then the following two statements are equivalent.*

(i) *The system (2) admits a unique solution on $[0, \infty)$ for any initial state x_0.*

(ii) *For the system (2), the smooth continuation from every initial state $x_0 \in \mathbb{R}^n$ is possible only in one of the m modes. In other words, the smooth continuation is possible only in one of the sets*

$$\left\{x \in \mathbb{R}^n : h_i^1(x) \geq 0, \ h_i^2(x) \geq 0, \ \ldots, \ h_i^{p_i}(x) \geq 0\right\}, \quad i = 1, \ldots, m. \tag{5}$$

The exception is for cases where solutions in any two of the sets above are the same in some time interval.