Nonlinear Analysis 71 (2009) 6116–6121

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Maximum and antimaximum principles for some nonlocal diffusion operators

Jorge García-Melián [a,b,*], Julio D. Rossi [c]

[a] Departamento de Análisis Matemático, Universidad de La Laguna, C/. Astrofísico Francisco Sánchez s/n, 38271 La Laguna, Spain

[b] Instituto Universitario de Estudios Avanzados (IUdEA) en Física Atómica, Molecular y Fotónica, Facultad de Física, Universidad de La Laguna, C/. Astrofísico Francisco Sánchez s/n, 38203 La Laguna, Spain

[c] Dpto. de Matemáticas, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, 1428 Buenos Aires, Argentina

ARTICLE INFO

Article history:
Received 20 June 2008
Accepted 1 June 2009

Keywords:
Nonlocal diffusion
Maximum principle
Antimaximum principle
Principal eigenvalue

ABSTRACT

In this work we consider the maximum and antimaximum principles for the nonlocal Dirichlet problem

$$J * u - u + \lambda u + h = \int_{\mathbb{R}^N} J(x-y)u(y)\,dy - u(x) + \lambda u(x) + h(x) = 0$$

in a bounded domain Ω, with $u(x) = 0$ in $\mathbb{R}^N \setminus \Omega$. The kernel J in the convolution is assumed to be a continuous, compactly supported nonnegative function with unit integral. We prove that for $\lambda < \lambda_1(\Omega)$, the solution verifies $u > 0$ in $\overline{\Omega}$ if $h \in L^2(\Omega)$, $h \geq 0$, while for $\lambda > \lambda_1(\Omega)$, and λ close to $\lambda_1(\Omega)$, the solution verifies $u < 0$ in $\overline{\Omega}$, provided $\int_\Omega h(x)\phi(x)\,dx > 0$, $h \in L^\infty(\Omega)$. This last assumption is also shown to be optimal. The "Neumann" version of the problem is also analyzed.

1. Introduction

This note is mainly concerned with the validity of the maximum and antimaximum principles for the nonlocal inhomogeneous linear problem

$$\begin{cases} \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y)\,dy - u(x) + \lambda u(x) + h(x) = 0 & x \in \Omega, \\ u(x) = 0 & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^N$, h is a given function and λ is a real parameter. The operator $J * u - u = \int_{\mathbb{R}^N} J(x-y)u(y)\,dy - u(x)$ is a nonlocal diffusion operator that has been recently used to model several physical situations (see for example [1] in the context of biological models). Without further mention, we are always assuming that the kernel J is continuous, nonnegative, supported in the unit ball B of $\mathbb{R}^N$ and with unit integral. We also suppose that $J > 0$ in B and that $J(-x) = J(x)$ for every x. The condition $u(x) = 0$ in $\mathbb{R}^N \setminus \Omega$ is the nonlocal analogue to the usual Dirichlet boundary condition $u|_{\partial\Omega} = 0$ imposed when one considers the usual Laplacian as the diffusion operator, see [2].

The general problem

$$\begin{cases} (J * u)(x) - u(x) = f(x, u(x)) & x \in \Omega, \\ u(x) = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

* Corresponding author at: Departamento de Análisis Matemático, Universidad de La Laguna, C/. Astrofísico Francisco Sánchez s/n, 38271 La Laguna, Spain.

E-mail addresses: jjgarmel@ull.es (J. García-Melián), jrossi@dm.uba.ar (J.D. Rossi).

0362-546X/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.na.2009.06.004

and its parabolic version have been widely treated in the recent literature. In most of the references, $\Omega = \mathbb{R}^N$ so that the Dirichlet condition is not present. We quote for instance [3–11], devoted to travelling front type solutions to the parabolic problem when $\Omega = \mathbb{R}$, and [12–16], that study problem (1.2) with a logistic type, bistable or power-like nonlinearity. The particular instance of the parabolic problem in $\mathbb{R}^N$ when $f = 0$ is considered in [2,17], while the "Neumann" boundary condition for the same problem is treated in [18–20]. See also [21] for the appearance of convective terms and [22,23] for interesting features in other related nonlocal problems. We also mention the paper [24], where some logistic equations and systems of Lotka–Volterra type are studied, and interesting biological conclusions are obtained.

Problem (1.1) is the nonlocal analogue of

$$\begin{cases} \Delta u(x) + \lambda u(x) + h(x) = 0 & x \in \Omega, \\ u(x) = 0 & x \in \partial\Omega. \end{cases} \tag{1.3}$$

One can see this analogy considering the functional whose critical points are solutions to (1.1) and performing a first order Taylor expansion; we refer to [12] for details.

Since the maximum and antimaximum principles have shown to be powerful tools when analyzing nonlinear elliptic problems related to (1.3), we want to analyze them in the context of the nonlocal problem (1.1).

Recall that the maximum principle is well known for problem (1.3) in the following form: if $h \geq 0$ and $\lambda < \sigma_1(\Omega)$, where $\sigma_1(\Omega)$ denotes the first eigenvalue of the Dirichlet Laplacian in Ω, then $u > 0$ in Ω and $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$, where ν stands for the outward unit normal to $\partial\Omega$. Moreover, the condition $\lambda < \lambda_1(\Omega)$ is known to be also necessary (see [25–27] and also [28] for a version involving the p-Laplacian).

With respect to the antimaximum principle, it has been proved that for $h \in L^p(\Omega)$, $p > N$, such that $\int_\Omega h(x)\psi(x)\,\mathrm{d}x > 0$, with $\psi > 0$ an eigenfunction associated to $\sigma_1(\Omega)$, there exists $\delta > 0$ such that $u < 0$ in Ω and $\frac{\partial u}{\partial \nu} > 0$ on $\partial\Omega$ for $\sigma_1(\Omega) < \lambda < \sigma_1(\Omega) + \delta$. See [29,30], and also [31] for a quasilinear version. It was later proved in [32] that the condition $h \in L^p(\Omega)$, $p > N$ is necessary.

Our main objective in this work is to show that a version of the maximum and antimaximum principles remains valid for problem (1.1).

A word on the notion of solution to (1.1): by a solution we mean a function $u \in L^1(\Omega)$ which verifies (1.1) almost everywhere. Although in most places we are dealing with $h \in L^2(\Omega)$, which forces $u \in L^2(\Omega)$. However, we remark that with $h \in L^\infty(\Omega)$ it always follows that $u \in L^\infty(\Omega)$, and we are requiring this extra regularity for the validity of the antimaximum principle. It will be also shown that this condition is not superfluous.

An important role in what follows will be played by the principal eigenvalue $\lambda_1(\Omega)$ of the problem

$$\begin{cases} (J * u)(x) - u(x) + \lambda u(x) = 0 & x \in \Omega, \\ u(x) = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.4}$$

whose existence and properties will be briefly considered in Section 2. Now, we only quote that $\lambda_1(\Omega)$ is positive and less than one, and it has an associated eigenfunction $\phi \in C(\overline{\Omega})$ which verifies $\phi > 0$ in $\overline{\Omega}$ (let us mention in passing that ϕ has a jump discontinuity across $\partial\Omega$, see [33]).

We now come to the statement of the maximum principle. Let us mention that problem (1.1) has a unique solution for every $h \in L^2(\Omega)$ provided $\lambda < \lambda_1(\Omega)$ or $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega) + \varepsilon)$, if ε is small enough (see Remark 1 in Section 2).

Theorem 1. *Let $h \in L^2(\Omega)$ be such that $h \geq 0$, $h \not\equiv 0$, and let $u \in L^2(\Omega)$ be the solution to (1.1) with $\lambda < \lambda_1(\Omega)$. Then $u > 0$ in $\overline{\Omega}$.*

We remark that the condition $u > 0$ in $\overline{\Omega}$ means in this context that $\operatorname{ess\,inf} u > 0$, since for $h \in L^2(\Omega)$ we only have $u \in L^2(\Omega)$.

Next, we consider the antimaximum principle.

Theorem 2. *Let $h \in L^\infty(\Omega)$ verify*

$$\int_\Omega h(x)\phi(x)\mathrm{d}x > 0.$$

Then there exists $\varepsilon = \varepsilon(h) > 0$ such that for $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega) + \varepsilon)$ the unique solution to (1.1) verifies $u < 0$ in $\overline{\Omega}$.

We also consider the question of optimality of the hypothesis $h \in L^\infty(\Omega)$. We stress that if $h \notin L^\infty(\Omega)$ then $u \notin L^\infty(\Omega)$, and this provokes the failure of the antimaximum principle.

Theorem 3. *There exists a small $\varepsilon > 0$ and $h \in L^2(\Omega)$, $h \notin L^\infty(\Omega)$ with*

$$\int_\Omega h(x)\phi(x)\,\mathrm{d}x > 0$$

such that the unique solution to (1.1) for $\lambda \in (\lambda_1(\Omega), \lambda_1(\Omega) + \varepsilon)$, is positive somewhere in Ω.