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Quasilinear elliptic equations involving the N-Laplacian with critical exponential growth in $\mathbb{R}^N$ ☆

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ABSTRACT

This paper shows the existence of nontrivial solutions for the quasilinear equations of the form

$$-\Delta_N u + V(x)|u|^{N-2}u - \Delta_N(u^2)u = h(u) \quad \text{in } \mathbb{R}^N,$$

where Δ_N is the N-Laplacian operator, V is a continuous function bounded from below away from zero and $h(u)$ is a continuous function having critical exponential growth.

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1. Introduction

This paper is concerned with the quasilinear equations of the form

$$-L_N u + V(x)|u|^{N-2}u = h(u) \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where

$$L_N u := \Delta_N u + \Delta_N(u^2)u,$$

and $\Delta_N u := \mathrm{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator with $N \geq 2$.

There has been recently a good amount of work on the quasilinear equations

$$-\Delta u + V(x)u - \Delta(|u|^2)u = h(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

Solutions of this equation are related to the existence of standing wave solutions for quasilinear Schrödinger equations of the form

$$iz_t = -\Delta z + V(x)z - h(|z|^2)z - k\Delta g(|z|^2)g'(|z|^2)z, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where V is a given potential, k is a real constant, h and g are real functions. The related semilinear equations for $k = 0$ have been intensively studied (see e.g. [1–7], as well as their references). For $k \neq 0$, we refer to the papers [8–19]. Quasilinear equations of the form (1.3) appear more naturally in mathematical physics and have been derived as models

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of several physical phenomena corresponding to various types of g. In the case $g(s) = (1+s)^{1/2}$, Eq. (1.3) models the self-channeling of a high-power ultrashort laser in matter (see [20,17]). The case $g(s) = s$ and $N = 1$ is treated in [18]. Poppenberg, Schmit and Wang in [12] studied the existence of a positive ground state solution for the quasilinear Schrödinger equation $-u'' + V(x)u - (u^2)''u = \theta |u|^{p-1}u$ in $\mathbb{R}$. In [19], Ambrosetti–Wang considered the quasilinear problem $-u'' + [1+\varepsilon a(x)]u - k[1+\varepsilon b(x)](u^2)'' = [1+\varepsilon c(x)]u^q$, where $q > 1$, k, ε are real numbers and a, b, c are real-valued functions belonging to a certain class S. The authors used a variational method, together with a perturbation technique to prove that there exists $k_0 > 0$ such that for $k > -k_0$, the equation has a positive solution $u \in H^1(\mathbb{R})$, provided that $|\varepsilon|$ is sufficiently small. In [9], by a change of variables the quasilinear problem was transformed to a semilinear one and an Orlicz space framework was used, as the working space, and they were able to prove the existence of positive solutions of (1.2) by the Mountain Pass theorem. The same method of changing of variables was used in [11], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space and they studied a different class of nonlinearities. In [10], the existence of both one-sign and nodal ground states soliton type solutions was established by the Nehari method.

Motivated by the quasilinear Schrödinger equation (1.2), Alves et al. in [21] considered the following equations

$$Lu + V(x)|u|^{p-2}u = |u|^{q-2}u + g(x), \quad \text{in } \mathbb{R}, \tag{1.4}$$

where

$$Lu = -[|u'|^{p-2}u']' - k_0\{[(|u|^\beta)']^{p-2}(|u|^\beta)'\}'|u|^{\beta-2}u,$$

$k_0 > 0, \beta > 1, p > 1, q \geq p\beta (q > p), g \in L^s(\mathbb{R})$ for some $s \geq 1$. By applying the variational method, they proved the existence of two positive solutions for (1.4). We also refer to the recent work of Severo in [22], where he considered the equation

$$-L_p u + V(x)|u|^{p-2}u = h(u) \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where

$$L_p u := \Delta_p u + \Delta_p(u^2)u, \quad 1 < p \leq N,$$

and h has subcritical growth. By using the Sobolev space $W^{1,p}(\mathbb{R}^N)$, he proved the existence of solutions from the results given by do Ó and Medeiros [23] when $1 < p \leq N$.

The main purpose of the present paper is to study the problem (1.1) when $p = N$ and h satisfies the critical growth conditions.

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 2)$, the Trudinger–Moser inequality [24] asserts that

$$\exp(\alpha |u|^{\frac{N}{N-1}}) \in L^1(\Omega), \quad \forall u \in W_0^{1,N}(\Omega),\ \forall \alpha > 0,$$

and that there exists a constant $C(N)$, such that

$$\sup_{\|u\|_{W_0^{1,N}(\Omega)} \leq 1} \int_\Omega \exp(\alpha |u|^{\frac{N}{N-1}}) \mathrm{d}x \leq C(N)|\Omega|, \quad \text{if } \alpha \leq \alpha_N,$$

where $\alpha_N = N\omega_{N-1}^{\frac{1}{N-1}}$ and ω_{N-1} is the $(N-1)$-dimensional measure of the $(N-1)$ sphere. Subsequently, do Ó in [25] proved a version of Trudinger–Moser inequality in whole space, namely

$$\exp(\alpha |u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u) \in L^1(\mathbb{R}^N), \quad \forall u \in W_0^{1,N}(\mathbb{R}^N),\ \forall \alpha > 0.$$

Moreover, if $\alpha < \alpha_N$ and $\|u\|_{L^N} \leq M$, there exists a constant $C = C(N, M, \alpha)$ such that

$$\sup_{\|\nabla u\|_N^N \leq 1} \int_{\mathbb{R}^N} [\exp(\alpha |u|^{\frac{N}{N-1}}) - S_{N-2}(\alpha, u)] \mathrm{d}x \leq C(N, M, \alpha), \tag{1.6}$$

where $S_{N-2}(\alpha, u) = \sum_{k=0}^{N-2} \frac{\alpha^k}{k!} |u|^{\frac{Nk}{N-1}}$.

In this paper we assume that h satisfies the following critical growth condition:

$(c)_{\alpha_0}$ There exists $\alpha_0 > 0$ such that

$$\lim_{s\to\infty} \frac{|h(s)|}{\exp(\alpha |s|^{\frac{2N}{N-1}})} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases}$$

We believe that the exponential growth mentioned above is the critical growth for problem (1.1) when $N = p$, since when $N = 2$ whose critical exponent is

$(\tilde{c})_{\alpha_0}$ There exists $\alpha_0 > 0$ such that $(c)_{\alpha_0}$

$$\lim_{s\to\infty} \frac{|h(s)|}{\exp(\alpha |s|^4)} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0, \end{cases}$$

(see [13]).

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