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Periodic solutions for generalized high-order neutral differential equation in the critical case[☆]

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ABSTRACT

By applying Mawhin's continuation theory and some new inequalities, we obtain sufficient conditions for the existence of periodic solutions for a generalized high-order neutral differential equation in the critical case. Moreover, an example is given to illustrate the results.

1. Introduction

Consider a generalized high-order neutral differential equation in the following form

$$(\varphi_p(x(t)-cx(t-\tau))^{(l)})^{(n-l)}=F(t,x(t),x'(t),\dots,x^{(l-1)}(t)), \tag{1.1}$$

where $\varphi_p:\mathbb{R}\to\mathbb{R}$ is given by $\varphi_p(s)=|s|^{p-2}s$ with $p\geq 2$ is a constant, F is a continuous function defined on $\mathbb{R}^l$ and is periodic to t with $F(t,\cdot,\dots,\cdot)=F(t+2\pi,\cdot,\dots,\cdot)$, $F(t,a,0,\dots,0)\not\equiv 0$ for all $a\in\mathbb{R}$, τ is a constant.

Complicated behavior of models for technical applications is often described by nonlinear high-order differential equations [1], for example, the Lorenz model of a simplified hydrodynamic flow, the dynamo model of erratic inversion of the earth's magnetic field, etc. Oftentimes high-order equations are a result of combinations of lower-order equations. Due to its obvious complexity, studies on high-order differential equation are rather infrequent, especially on high-order delay differential equation. Most of the results on high-order delay differential equation are concentrated in the few years. In [2], Cheng and Ren present the existence of periodic solutions for a fourth-order Rayleigh type p-Laplacian delay differential equation as follows

$$(\varphi_p(x(t)''))''+f(t,x'(t-\sigma(t)))+g(t,x(t-\tau(t)))=e(t). \tag{1.2}$$

In [3], Pan studies the nth-order differential equation

$$x^{(n)}(t)=\sum_{i=1}^{n-1}b_ix^{(i)}(t)+f(t,x(t),x(t-\tau_1(t)),\dots,x(t-\tau_m(t)))+p(t), \tag{1.3}$$

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and obtain the existence of periodic solutions for Eq. (1.3). Afterwards, Ren and Cheng [4] obtain sufficient conditions for the existence of periodic solutions for a general high-order delay differential equation

$$x^{(n)}(t) = F(t, x(t), x(t-\tau(t)), x'(t), \ldots, x^{(n-1)}(t)). \tag{1.4}$$

In [5], Li and Lu consider the following high-order p-Laplacian differential equation

$$(\varphi_p(y^{(m)}(t)))^{(m)} = f(y(t))y'(t) + h(y(t)) + \beta(t)g(y(t-\tau(t))) + e(t), \tag{1.5}$$

and by using the theory of Fourier series, Bernoulli number theory and continuation theorem of coincidence degree theory, they get the existence of periodic solutions for Eq. (1.5). In [6], Wang and Lu investigate the existence for the high-order neutral functional differential equation with distributed delay

$$(x(t) - cx(t-\sigma))^{(n)} + f(x(t))x'(t) + g\left(\int_{-r}^{0} x(t+s)\mathrm{d}\alpha(s)\right) = p(t). \tag{1.6}$$

Inspired by these results, we consider a generalized high-order neutral differential equation (1.1). In [7], Ren, Cheung and Cheng have researched on periodic solution for (1.1) in the case $|c| \neq 1$. In the paper, we shall move on the research of (1.1) in the more complicated case, i.e, in the critical case $|c| = 1$. By citing some results on Lu [8,9] and Zhang [10], we first carry out further results on the neutral operator in the critical case and then by applying Mawhin's continuation theorem we obtain sufficient conditions for the existence of periodic solutions for Eq. (1.1). Our results are new and our methods are different from the above works. Meanwhile, an example is given to illustrate our results.

Throughout this paper, we will denote by Z the set of integers, Z_1 the set of odd integers, Z_2 the set of even integers, N the set of positive integers, N_1 the set of odd positive integers and N_2 the set of even positive integers. Let $C_{2\pi}^1 = \{x : x \in C^1(\mathbb{R}, \mathbb{R}), x(t+2\pi) \equiv x(t)\}$ with the norm $|\varphi|_{C_{2\pi}^1} = \{\max_{t\in[0,2\pi]} |\varphi(t)|, \max_{t\in[0,2\pi]} |\varphi'(t)|\}$, $C_{2\pi} = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t+2\pi) \equiv x(t)\}$ with the norm $|\varphi|_0 = \max_{t\in[0,2\pi]} |\varphi(t)|$, $C_{2\pi}^0 = \{x : x \in C_{2\pi}, \int_0^{2\pi} x(s)\mathrm{d}s = 0\}$, $C_{2\pi}^- = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t+\pi) \equiv -x(t)\}$, $C_{2\pi}^+ = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t+\pi) \equiv x(t)\}$ and $C_{2\pi}^{+,0} = \{x : x \in C_{2\pi}^+, \int_0^{2\pi} x(s)\mathrm{d}s = 0\}$ equipped with the norm $|\cdot|_0$, $L^2 = \{x : \mathbb{R} \to \mathbb{R}$ is 2π periodic and its restriction to $[0, 2\pi]$ belongs to $L^2([0,\pi])\}$, under the norm $|\varphi|_2 = \left(\int_0^{2\pi} |\varphi(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}$, $L^{2-} = \{x : x \in L^2, x(t+\pi) \equiv -x(t)\}$ and $L^{2+} = \{x : x \in L^2, x(t+\pi) \equiv x(t)\}$ with the norm $|\cdot|_2$. Clearly, $C_{2\pi}^1, C_{2\pi}, C_{2\pi}^+, C_{2\pi}^0, C_{2\pi}^{+,0}, L^2, L^{2-}$ and L^{2+} are all Banach space. We also denote $\bar{h} = \frac{1}{2\pi}\int_0^{2\pi} h(s)\mathrm{d}s$, $\forall\, h \in L^2$.

2. Preparation

Let X and Y be real Banach spaces and $L : D(L) \subset X \to Y$ be a Fredholm operator with index zero, here $D(L)$ denotes the domain of L. This means that $\operatorname{Im} L$ is closed in Y and $\dim \operatorname{Ker} L = \dim(Y/\operatorname{Im} L) < +\infty$. Consider supplementary subspaces X_1, Y_1, of X, Y respectively, such that $X = \operatorname{Ker} L \oplus X_1$, $Y = \operatorname{Im} L \oplus Y_1$, and let $P : X \to \operatorname{Ker} L$ and $Q : Y \to Y_1$ denote the natural projections. Clearly, $\operatorname{Ker} L \cap (D(L) \cap X_1) = \{0\}$, thus the restriction $L_P := L|_{D(L)\cap X_1}$ is invertible. Let K denote the inverse of L_P.

Let Ω be an open bounded subset of X with $D(L) \cap \Omega \neq \emptyset$. A map $N : \overline{\Omega} \to Y$ is said to be L-compact in $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and the operator $K(I-Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.1 (*Gaines and Mawhin [11]*). *Suppose that X and Y are two Banach spaces, and $L : D(L) \subset X \to Y$ is a Fredholm operator with index zero. Furthermore, $\Omega \subset X$ is an open bounded set and $N : \overline{\Omega} \to Y$ is L-compact on $\overline{\Omega}$. Assume that the following conditions hold:*

(1) $Lx \neq \lambda Nx$, $\forall\, x \in \partial\Omega \cap D(L)$, $\lambda \in (0, 1)$;
(2) $Nx \notin \operatorname{Im} L$, $\forall\, x \in \partial\Omega \cap \operatorname{Ker} L$;
(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is an isomorphism,*

then the equation $Lx = Nx$ has a solution in $\overline{\Omega} \cap D(L)$.

Lemma 2.2 (*[10]*). *If $\omega \in C^1(\mathbb{R}, \mathbb{R})$ and $\omega(0) = \omega(T) = 0$, then*

$$\int_0^T |\omega(t)|^p\mathrm{d}t \leq \left(\frac{T}{\pi_p}\right)^p \int_0^T |\omega'(t)|^p\mathrm{d}t$$

where p is a fixed real number with $p > 1$, and $\pi_p = 2\int_0^{(p-1)/p} \frac{\mathrm{d}s}{(1-\frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$.

First, we define operators A in the following form:

$$A : C_{2\pi} \to C_{2\pi}, \qquad (Ax)(t) = x(t) - cx(t-\tau).$$