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Shape optimization of stationary Navier–Stokes equation over classes of convex domains

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ABSTRACT

In this work, we obtain some properties for the family of some convex domains. Based on these, we prove the existence of solutions of some shape optimization for stationary Navier–Stokes equations.

1. Introduction

Let $U_R = U(0, R)$ be an open ball with center origin and radius R in $\mathbb{R}^N$, $N = 2, 3$, and $\mathcal{O}_c$ be a family of open convex domains included in U_R which will be precised later. Consider the following stationary Navier–Stokes equation in domain $\Omega \subset \mathbb{R}^N$:

$$\begin{cases} -\nu\triangle u + (u\cdot\nabla)u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\nu > 0$ is the viscosity constant of the fluid and $f \in (L^2(U_R))^N$ is a given function, u denotes the velocity while p denotes the pressure, and $\Omega \in \mathcal{O}_c$.

For each open subset $\omega \subset \mathbb{R}^N$, we denote $C^\infty_{0,\sigma}(\omega) = \{u \in (C^\infty_0(\omega))^N; \operatorname{div} u = 0\}$ and $H^1_{0,\sigma}(\omega) = \overline{C^\infty_{0,\sigma}(\omega)}^{\|\cdot\|_{H^1}}$, the completion of $C^\infty_{0,\sigma}(\omega)$ in the norm of $(H^1(\omega))^N$.

We say that u is a weak solution of (1) if $u \in H^1_{0,\sigma}(\Omega)$ and

$$\int_\Omega \nabla u\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega (u\cdot\nabla)u\cdot\varphi\,\mathrm{d}x = \int_\Omega f\cdot\varphi\,\mathrm{d}x, \tag{2}$$

for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$.

It is well known that (see [1,2]) for each $\Omega \in \mathcal{O}_c$, Eq. (1) has at least one weak solution, moreover, there exists a positive constant $\mathrm{C}(\nu, R)$ depending only on the viscosity constant ν and the radius R of set U_R such that if

$$\|f\|_{(L^2(U))^N} < \mathrm{C}(\nu, R) \tag{3}$$

then the weak solution of (1) corresponding to each Ω is unique. We shall assume in this paper that (3) holds.

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In this paper, we shall study the following shape optimization problem

$$(P) \qquad \inf_{\Omega\in\mathcal{O}_c} \int_\Omega F(x, u_\Omega, \nabla u_\Omega)\,\mathrm{d}x$$

where u_Ω is a weak solution of (1) corresponding to $\Omega \in \mathcal{O}_c$, $F(x,\xi,\eta) : U_R \times \mathbb{R}^N \times \mathbb{R}^{N\times N} \to \mathbb{R}^+$ is continuous and satisfies that there exists a positive constant s such that

$$|F(x,\xi,\eta)| \le s(1 + |\xi|^2_{\mathbb{R}^N} + |\eta|^2_{\mathbb{R}^{N\times N}}) \tag{4}$$

for all $(x,\xi,\eta) \in U_R \times \mathbb{R}^N \times \mathbb{R}^{N\times N}$.

Now we define

$$\mathcal{O}_c = \{\Omega \subset U_R;\ \Omega \text{ is convex domain in } U_R \text{ with } \mathscr{L}^N(\Omega) \ge c\}$$

where $\mathscr{L}^N(\Omega)$ is the Lebesgue measure of Ω and c is a fixed positive constant.

The topology on $\mathcal{O}_c$ is induced from the Hausdorff–Pompeiu distance between the complementary sets, i.e.,

$$\rho(\Omega_1,\Omega_2) = \max\{\sup_{x\in\overline{U_R}\setminus\Omega_1} d(x, \overline{U_R}\setminus\Omega_2),\ \sup_{y\in\overline{U_R}\setminus\Omega_2} d(\overline{U_R}\setminus\Omega_1, y)\}, \quad \forall\ \Omega_1,\Omega_2 \in \mathcal{O}_c, \tag{5}$$

where $d(\cdot,\cdot)$ denotes the Euclidean metric in $\mathbb{R}^N$. We denote by Hlim, the limit in the sense of (5).

In fact, the similar families $\mathcal{O}_c$ have been discussed in [3–7]. But we will obtain the results by some different ways in this paper.

In this work, we obtain the following main result on the family $\mathcal{O}_c$:

Theorem 2.1. *If $\{\Omega_m\}_{m=1}^\infty \subset \mathcal{O}_c$, then there exists a subsequence $\{\Omega_{m_k}\}_{k=1}^\infty$ of $\{\Omega_m\}_{m=1}^\infty$ such that*

$$\operatorname*{Hlim}_{k\to\infty} \Omega_{m_k} = \Omega \quad \text{and} \quad \Omega \in \mathcal{O}_c.$$

i.e., $(\mathcal{O}_c, \rho)$ is a compact metric space.

Based on these, we obtain the existence of the optimal solutions for problem (P):

Theorem 3.1. *The shape optimization problem (P) has at least one solution.*

2. Some properties related to the family $\mathcal{O}_c$

In this section, we assume that $k > 0$ is an arbitrary integral; $d(A, B)$, $A, B \subset \mathbb{R}^k$, denotes the distance between sets A and B in $\mathbb{R}^k$, especially, we denote $d(\{a\}, B) = d(a, B)$; and we shall use the following notations:

$$\delta(K_1,K_2) = \max\{\sup_{x\in K_1} d(x,K_2), \sup_{y\in K_2} d(K_1,y)\},$$

where K_1 and K_2 are compact subsets in $\mathbb{R}^k$; $U(x, r)$ denotes the open ball in $\mathbb{R}^k$ with center x and radius r; $[x_0, x_1] = \{tx_0 + (1-t)x_1 \in \mathbb{R}^k;\ 0 \le t \le 1\}$ is a closed segment in $\mathbb{R}^k$ with two extremal points x_0, x_1; $L_i = L(a_0,\ldots,a_{i-1},\hat{a}_i,a_{i+1},\ldots,a_t) = L(a_0,\ldots,a_{i-1},a_{i+1},\ldots,a_t)$ denotes the hyperplane spanned by t vectors $a_i \in \mathbb{R}^k$, $i = 0,\ldots,i-1,i+1,\ldots,t$.

The following definitions and results were given and proved in [8,9,3,10–12], which will be used in this paper.

Definition 2.1. A set $A = \{\gamma_0,\gamma_1,\ldots,\gamma_t\}$ of $t + 1$ points in $\mathbb{R}^k$, is *geometrically independent* means that no hyperplane of dimension $t - 1$ contains all the points.

Definition 2.2. Let $\{\gamma_0,\gamma_1,\ldots,\gamma_t\}$ be a set of geometrically independent points in $\mathbb{R}^k$. The *t-dimensional geometric simplex* or *t-simplex*, σ^t, spanned by $\{\gamma_0,\gamma_1,\ldots,\gamma_t\}$ is the set of all points $x \in \mathbb{R}^k$ for which there exist nonnegative real numbers $\lambda_0,\ldots,\lambda_t$ such that

$$x = \sum_{i=0}^t \lambda_i\gamma_i, \quad \sum_{i=0}^t \lambda_i = 1.$$

The numbers $\lambda_0,\ldots,\lambda_t$ are the *barycentric coordinates*. The points $\{\gamma_0,\gamma_1,\ldots,\gamma_t\}$ are the *vertices* of σ^t. The set of all points x in σ^t with all barycentric coordinates positive is called the open geometric t-simplex spanned by $\{\gamma_0,\gamma_1,\ldots,\gamma_t\}$.

Definition 2.3. A simplex σ^{k-1} is a $(k-1)$-*face* of a simplex σ^k means that each vertex of σ^{k-1} is a vertex of σ^k.

Definition 2.4. Let σ^k be a k-dimensional geometric simplex and $U(x, r)$ be an open ball with center x and radius r in $\mathbb{R}^k$ we call $U(x, r)$ an *interior contact ball* of σ^k if $U(x, r) \subset \sigma^k$ and each $(k-1)$-face of σ^k tangent to the ball $U(x, r)$, i.e., $\mathrm{Card}(\sigma^{k-1} \cap \overline{U(x,r)}) = 1$. Here $\mathrm{Card}(\sigma^{k-1} \cap \overline{U(x,r)})$ denotes the cardinality of the set $\sigma^{k-1} \cap \overline{U(x,r)}$.

Lemma 2.1. *Let A, A_n, $n = 1, 2, \ldots$, be compact subsets in $\mathbb{R}^k$ such that $\delta(A_n, A) \to 0$, then A is the set of all accumulation points of the sequences $\{x_n\}$ such that $x_n \in A_n$ for each n.*