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Stein iterations for the coupled discrete-time Riccati equations

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ABSTRACT

We consider a set of discrete-time coupled algebraic Riccati equations that arise in quadratic optimal control of Markovian jump linear systems. Two iterations for computing a symmetric (maximal) solution of this system are investigated. We construct sequences of the solutions of the decoupled Stein equations and show that these sequences converge to a solution of the considered system. Numerical experiments are given.

1. Introduction

We consider the problem for computing a Hermitian solution to the following set of coupled algebraic Riccati equations:

$$\mathcal{P}_k(\mathbf{X}) := A_k^{\mathrm{T}}\mathcal{E}_k(\mathbf{X})A_k - (A_k^{\mathrm{T}}\mathcal{E}_k(\mathbf{X})B_k + L_k)\left(D_k^{\mathrm{T}}D_k + B_k^{\mathrm{T}}\mathcal{E}_k(\mathbf{X})B_k\right)^{-1}(L_k^{\mathrm{T}} + B_k^{\mathrm{T}}\mathcal{E}_k(\mathbf{X})A_k) + C_k^{\mathrm{T}}C_k = X_k \quad (1)$$

where $\mathcal{E}(\mathbf{X}) = (\mathcal{E}_1(\mathbf{X}), \dots, \mathcal{E}_N(\mathbf{X}))$ with $\mathbf{X} = (X_1, \dots, X_N)$ and

$$\mathcal{E}_k(\mathbf{X}) = \sum_{j=1}^{N} \lambda_{kj} X_j, \quad X_j \text{ is an } n \times n \text{ matrix, for } k = 1, \dots, N.$$

In (1), A_k, B_k, L_k, D_k and C_k $(k = 1, \dots, N)$ are given real matrices of appropriate dimensions. Normally the matrix $\begin{pmatrix} Q_k & L_k \\ L_k^{\mathrm{T}} & R_k \end{pmatrix}$ is assumed to be positive semidefinite with $Q_k = C_k^{\mathrm{T}}C_k$ and $R_k = D_k^{\mathrm{T}}D_k$.

A set of Eqs. (1) arises in the optimal control of discrete-time jump linear systems defined by

$$x_{i+1} = A(r_i)x_i + B(r_i)u_i \quad (2)$$

where $A(r_i) = A_k, B(r_i) = B_k$ when $r_i = k, k = 1, \dots, N$. The process r_i is a finite-state discrete-time Markov chain with transition probabilities

$$\mathbf{Prob}\{r_{i+1} = j | r_i = k\} = p_{kj}, \quad \text{with } k, j = 1, \dots, N.$$

Here $p_{kj} \geq 0$ for all k, j. It is necessary to minimize a corresponding functional [1,2] subject to (2).

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It has been shown in [3] that the solution of this problem is associated with the existence of a solution $\mathbf{P} = (P_1, \ldots, P_N)$ and $P_k \geq 0$, $k = 1, \ldots, N$ to (1). If such a solution $\mathbf{P}$ exists, it can be shown [3] that the optimal control law for the above problem is given by the feedback control law

$$u(k) = F_{\theta(k)} x(k)$$

where $F = (F_1, \ldots, F_N)$ is given by

$$F_k = F_{k,\mathbf{P}} = -(B_k^T \mathcal{E}_k(\mathbf{P}) B_k + D_k^T D_k)^{-1} B_k^T \mathcal{E}_k(\mathbf{P}) A_k.$$

Thus, a problem to find an optimal control amounts to solving a set of Eq. (1). Conditions for the existence of the maximal solution were presented in [3] in terms of the concept of mean square stabilizability. Characterization of the maximal solution in terms of a linear matrix inequalities (LMI) optimization problem has been presented in [4]. Sufficient conditions for the existence of the maximal solution for (1) have been derived in [2, Theorem 1]. The approach for proving this theorem defines a numerical algorithm for computing a maximal solution of (1) (**Algorithm CM** below). Costa and Aya [1] have shown how to apply temporal difference methods for obtaining a solution $\mathbf{P}$ to (1).

Here we introduce a new approach to solve a set of Riccati equations (1) and we consider two algorithms for finding the maximal solution to (1). These algorithms are derived after some matrix manipulations on system (1) and they are presented in terms of decoupled algebraic Stein equations. The new algorithms are based on the nonlinear Jacobi/Gauss–Seidel techniques. We are going to prove the convergence properties to the proposed iterations under new assumptions. The presented algorithms fill a gap that existed in the numerical solution of the discrete-time jump parameter optimal control problem as compared to the corresponding continuous-time problem.

The notation $\mathcal{H}^n$ stands for the linear space of symmetric matrices of size n over the field of real numbers. For any $X, Y \in \mathcal{H}^n$, we write $X > Y$ or $X \geq Y$ if $X - Y$ is positive definite or $X - Y$ is positive semidefinite. The eigenvalues for any square real $n \times n$ matrix A will be denoted by $\lambda_s(A)$, $s = 1, \ldots, n$ and $\rho(A)$ stands for the spectral radius of A. A matrix A is said to be asymptotically d-stable if all eigenvalues of A lie in the open unit disk, i.e. $|\lambda_s(A)| < 1$ for $s = 1, \ldots, n$, and almost asymptotically d-stable if $|\lambda_s(A)| \leq 1$ for $s = 1, \ldots, n$. The notations $\mathbf{X} \in \mathcal{H}^n$ and the inequality $\mathbf{Y} \geq \mathbf{Z}$ mean that for $k = 1, \ldots, N$, $X_k \in \mathcal{H}^n$ and $Y_k \geq Z_k$, respectively. The linear space $\mathcal{H}^n$ is a Hilbert space with the Frobenius inner product $\langle X, Y \rangle = trace(XY)$. For a linear operator $\mathcal{L}$ on $\mathcal{H}^n$, let $r_\sigma(\mathcal{L})$ denote the spectral radius.

We introduce notations

$$S_{k,\mathbf{Z}} = A_k^T \mathcal{E}_k(\mathbf{Z}) B_k + L_k; \qquad K_{k,\mathbf{Z}} = R_k + B_k^T \mathcal{E}_k(\mathbf{Z}) B_k; \qquad F_{k,\mathbf{Z}} = -K_{k,\mathbf{Z}}^{-1} S_{k,\mathbf{Z}}^T, \quad \text{and}$$

$$\begin{aligned} T_{k,\mathbf{Z}} &= Q_k + F_{k,\mathbf{Z}}^T L_k^T + L_k F_{k,\mathbf{Z}} + F_{k,\mathbf{Z}}^T R_k F_{k,\mathbf{Z}} \\ &= \begin{pmatrix} I & F_{k,\mathbf{Z}}^T \end{pmatrix} \begin{pmatrix} Q_k & L_k \\ L_k^T & R_k \end{pmatrix} \begin{pmatrix} I \\ F_{k,\mathbf{Z}} \end{pmatrix} \end{aligned} \tag{3}$$

and present Eqs. (1) as follows:

$$A_k^T \mathcal{E}_k(\mathbf{X}) A_k + Q_k - S_{k,\mathbf{X}} K_{k,\mathbf{X}}^{-1} S_{k,\mathbf{X}}^T = X_k, \quad k = 1, \ldots, N. \tag{4}$$

Let us define the matrix functions

$$\begin{aligned} \mathcal{P}_k(\mathbf{X}) &= A_k^T \mathcal{E}_k(\mathbf{X}) A_k + Q_k - S_{k,\mathbf{X}} K_{k,\mathbf{X}}^{-1} S_{k,\mathbf{X}}^T, \quad k = 1, \ldots, N \\ &= A_k^T \mathcal{E}_k(\mathbf{X}) A_k + Q_k - F_{k,\mathbf{X}}^T K_{k,\mathbf{X}} F_{k,\mathbf{X}}. \end{aligned} \tag{5}$$

and we will study the system $\mathcal{P}_k(\mathbf{X}) = X_k$ for $k = 1, \ldots, N$. We start with some useful properties of $\mathcal{P}_k(\mathbf{X})$. For brevity we use the notation $\tilde{A}_{k,\mathbf{Z}} = A_k + B_k F_{k,\mathbf{Z}}$.

Lemma 1.1. *The following properties of $\mathcal{P}_k(\mathbf{X})$, $k = 1, \ldots, N$ hold:*

$$\mathcal{P}_{k,\mathbf{Z}}(\mathbf{Y}) = \tilde{A}_{k,\mathbf{Z}}^T \mathcal{E}_k(\mathbf{Y}) \tilde{A}_{k,\mathbf{Z}} + T_{k,\mathbf{Z}} - \left(F_{k,\mathbf{Y}}^T - F_{k,\mathbf{Z}}^T\right) K_{k,\mathbf{Y}} \left(F_{k,\mathbf{Y}} - F_{k,\mathbf{Z}}\right)$$

$$\mathcal{P}_{k,\mathbf{Z}}(\mathbf{Z}) - \mathcal{P}_{k,\mathbf{Z}}(\mathbf{Y}) = \tilde{A}_{k,\mathbf{Z}}^T \mathcal{E}_k(\mathbf{Z} - \mathbf{Y}) \tilde{A}_{k,\mathbf{Z}} + (F_{k,\mathbf{Y}}^T - F_{k,\mathbf{Z}}^T) K_{k,\mathbf{Y}} (F_{k,\mathbf{Y}} - F_{k,\mathbf{Z}}).$$

Proof. We write down

$$\begin{aligned} \mathcal{P}_k(\mathbf{Y}) &= A_k^T \mathcal{E}_k(\mathbf{Y}) A_k + Q_k - F_{k,\mathbf{Y}}^T K_{k,\mathbf{Y}} F_{k,\mathbf{Y}} \\ &= (A_k + B_k F_{k,\mathbf{Z}})^T \mathcal{E}_k(\mathbf{Y})(A_k + B_k F_{k,\mathbf{Z}}) + Q_k - F_{k,\mathbf{Y}}^T K_{k,\mathbf{Y}} F_{k,\mathbf{Y}} - A_k^T \mathcal{E}_k(\mathbf{Y}) B_k F_{k,\mathbf{Z}} - F_{k,\mathbf{Z}}^T B_k^T \mathcal{E}_k(\mathbf{Y}) A_k - F_{k,\mathbf{Z}}^T B_k^T \mathcal{E}_k(\mathbf{Y}) B_k F_{k,\mathbf{Z}} \\ &= \tilde{A}_{k,\mathbf{Z}}^T \mathcal{E}_k(\mathbf{Y}) \tilde{A}_{k,\mathbf{Z}} - F_{k,\mathbf{Y}}^T K_{k,\mathbf{Y}} F_{k,\mathbf{Y}} + T_{k,\mathbf{Z}} - F_{k,\mathbf{Z}}^T L_k^T - L_k F_{k,\mathbf{Z}} - F_{k,\mathbf{Z}}^T R_k F_{k,\mathbf{Z}} \\ &\quad - A_k^T \mathcal{E}_k(\mathbf{Y}) B_k F_{k,\mathbf{Z}} - F_{k,\mathbf{Z}}^T B_k^T \mathcal{E}_k(\mathbf{Y}) A_k - F_{k,\mathbf{Z}}^T B_k^T \mathcal{E}_k(\mathbf{Y}) B_k F_{k,\mathbf{Z}} \\ &= \tilde{A}_{k,\mathbf{Z}}^T \mathcal{E}_k(\mathbf{Y}) \tilde{A}_{k,\mathbf{Z}} - F_{k,\mathbf{Y}}^T K_{k,\mathbf{Y}} F_{k,\mathbf{Y}} + T_{k,Z} - F_{k,\mathbf{Z}}^T S_{k,\mathbf{Y}}^T - S_{k,\mathbf{Y}} F_{k,\mathbf{Z}} - F_{k,\mathbf{Z}}^T K_{k,\mathbf{Y}} F_{k,\mathbf{Z}} \\ &= \tilde{A}_{k,\mathbf{Z}}^T \mathcal{E}_k(\mathbf{Y}) \tilde{A}_{k,\mathbf{Z}} - F_{k,\mathbf{Y}}^T K_{k,\mathbf{Y}} F_{k,\mathbf{Y}} + T_{k,Z} + F_{k,\mathbf{Z}}^T K_{k,\mathbf{Y}} F_{k,\mathbf{Y}} + F_{k,\mathbf{Y}}^T K_{k,\mathbf{Y}} F_{k,\mathbf{Z}} - F_{k,\mathbf{Z}}^T K_{k,\mathbf{Y}} F_{k,\mathbf{Z}} \end{aligned}$$

$$\mathcal{P}_k(\mathbf{Y}) = \tilde{A}_{k,\mathbf{Z}}^T \mathcal{E}_k(\mathbf{Y}) \tilde{A}_{k,\mathbf{Z}} + T_{k,Z} - \left(F_{k,\mathbf{Y}}^T - F_{k,\mathbf{Z}}^T\right) K_{k,\mathbf{Y}} \left(F_{k,\mathbf{Y}} - F_{k,\mathbf{Z}}\right).$$